MATH221

quiz #3, 11/10/15

Total 100

Solutions

Show all work legibly. **Name**:______________________

1. (20) In the vector spave V of all real functions find a basis for span $\{\sin\ t,\ \sin\ 2t,\ \sin\ t\cos\ t\}$.

   Solution. Since $\sin\ 2t = 2\sin\ t\cos\ t$ the set $\{\sin\ t,\ \sin\ 2t,\ \sin\ t\cos\ t\}$ is linearly dependent, and span $\{\sin\ t,\ \sin\ 2t,\ \sin\ t\cos\ t\}$ = span $\{\sin\ t,\ \sin\ t\cos\ t\}$. Since $\sin\ t$ is not proportional to $\sin\ t\cos\ t$, the set $\{\sin\ t,\ \sin\ t\cos\ t\}$ is linearly independent.

   A basis is:

2. (20) Define $T\ :\ \mathbf{P}_2 \to \mathbf{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(1) \end{bmatrix}$.

   (a) (10) Describe Null $T = \{\mathbf{p}\ :\ \mathbf{p}(0) = 0,\ \text{and}\ \mathbf{p}'(1) = 0\}$.

   Solution. If $\mathbf{p}(x) = a_0 + a_1x + a_2x^2$, then

   $$0 = \mathbf{p}(0) = a_0,\ \text{and}\ 0 = \mathbf{p}'(1) = a_1 + 2a_2.$$

   Hence Null $T = \{\mathbf{p}\ :\ \mathbf{p}(x) = -2tx + tx^2\}$.

   Null $T =$

   (b) (10) Describe range of T.

   Solution. Let $\mathbf{p}_1(x) = 1$, and $\mathbf{p}_2(x) = x$. Note that

   $$T(\mathbf{p}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \text{and}\ T(\mathbf{p}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Hence range of T is $\mathbf{R}^2$.

   Range of T is

3. (20) Suppose $\mathbf{R}^4 = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. True or False? The vector set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

   Solution. Suppose the opposite, i.e. the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. In his case one of the vectors, say $\mathbf{v}_4$, is a linear combination of the other three vectors. This implies $\mathbf{R}^4 =$

Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there is a basis of $\mathbf{R}^4$ that contains less than 4 vectors. This contradiction completes the proof.

Mark one and explain.

▫ True ▫ False

4. (20) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a set of vectors in a vector space V so that every $\mathbf{v} \in \mathbf{V}$ has a unique representation as a linear combination of elements of $\mathcal{B}$. True or False? The vector set $\mathcal{B}$ is linearly independent.

   Solution. If $0 = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$, then $c_1 = c_2 = \ldots = c_n = 0$.

   Mark one and explain.

   ▫ True ▫ False

5. (20) Let H be a subspace of V, and $T : V \to W$ is a linear transformation betweet vector spaces V, and W.

   (a) (10) True or False? $T(H)$, the set of images of vectors in H, is a subspace of W.

   Solution. If $T(c_1\mathbf{h}_1 + c_2\mathbf{h}_2) = c_1T(\mathbf{h}_1) + c_2T(\mathbf{h}_2)$.

   Mark one and explain.

   ▫ True ▫ False

   (b) (10) True or False? $\dim T(H) \leq \dim H$.

   Solution. If $\{T(\mathbf{h}_1), \ldots, T(\mathbf{h}_k)\}$ is a basis for $T(H)$, then the vector set $\{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$ is linearly independent, hence $\dim T(H) = k \leq \dim H$.

   Mark one and explain.

   ▫ True ▫ False

6. (20) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Find rank $\mathbf{u}\mathbf{v}^T$.

   Solution. Since $\mathbf{u}\mathbf{v}^T = [4\mathbf{u}\ 5\mathbf{u}\ 6\mathbf{u}]$, rank $\mathbf{u}\mathbf{v}^T = 1$.

   rank $\mathbf{u}\mathbf{v}^T =$