Misc (Xiaowei Song)
Hölder's Inequality:
$\left|u^T v\right| \leq \left(\sum |u_i|^p\right)^{\frac{1}{p}} \left(\sum |v_i|^q\right)^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$
For

$$\Psi(z) = \log\left(\sum_{i=1}^{k} e^{-z_i}\right), z \in R^k$$

1. Ψ is convex (5 points)
Proof:
let $z_\lambda = \frac{1}{p}x + \frac{1}{q}y$,where $x, y \in R^k, p, q \in R^+$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = \frac{1}{p}, 1 - \lambda = \frac{1}{q}$, $0 < \lambda < 1$

$$\begin{aligned}
\Psi\left(\frac{1}{p}x + \frac{1}{q}y\right) = \Psi(z_\lambda) =& \log\left(\sum_{i=1}^{k} e^{-z_i}\right) \\
=& \log\left(\sum_{i=1}^{k} \exp\left[-\left(\frac{1}{p}x_i + \frac{1}{q}y_i\right)\right]\right) \\
=& \log\left(\sum_{i=1}^{k} \exp\left[-\left(\frac{1}{p}x_i\right)\right] \exp\left[-\left(\frac{1}{q}y\right)\right]\right) \\
=& \log\left(\sum_{i=1}^{k} \left[\exp\left(-x_i\right)\right]^{\frac{1}{p}} \left[\exp\left(-y_i\right)\right]^{\frac{1}{q}}\right) \\
=& \log\left(u^T v\right)
\end{aligned}$$

where

$$u = \begin{bmatrix} [\exp(-x_1)]^{\frac{1}{p}} \\ [\exp(-x_2)]^{\frac{1}{p}} \\ \vdots \\ [\exp(-x_k)]^{\frac{1}{p}} \end{bmatrix}, v = \begin{bmatrix} [\exp(-y_1)]^{\frac{1}{q}} \\ [\exp(-y_2)]^{\frac{1}{q}} \\ \vdots \\ [\exp(-y_k)]^{\frac{1}{q}} \end{bmatrix}$$

thus from Hölder's Inequality, we can have

$$\begin{aligned}
u^T v = \left|u^T v\right| \leq& \left(\sum |u_i|^p\right)^{\frac{1}{p}} \left(\sum |v_i|^q\right)^{\frac{1}{q}} \\
=& \left(\sum_{i=1}^{k} \left|\left[\exp\left(-x_i\right)\right]^{\frac{1}{p}}\right|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} \left|\left[\exp\left(-x_i\right)\right]^{\frac{1}{p}}\right|^p\right)^{\frac{1}{q}} \\
=& \left(\sum_{i=1}^{k} \exp\left(-x_i\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} \exp\left(-y_i\right)\right)^{\frac{1}{p}}
\end{aligned}$$

(where $u^T v > 0$ since $u_i > 0, v_i > 0, i = 1, \ldots, k$, thus $u^T v = \left|u^T v\right|$)
thus

$$\begin{aligned}
\Psi\left(\frac{1}{p}x + \frac{1}{q}y\right) = \Psi(z_\lambda) =& \log\left(u^T v\right) \\
\leq& \log\left(\sum_{i=1}^{k} \exp\left(-x_i\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} \exp\left(-y_i\right)\right)^{\frac{1}{p}} \\
=& \frac{1}{p}\log\left(\sum_{i=1}^{k} \exp\left(-x_i\right)\right) + \frac{1}{q}\log\left(\sum_{i=1}^{k} \exp\left(-y_i\right)\right) \\
=& \frac{1}{p}\Psi(x) + \frac{1}{q}\Psi(y)
\end{aligned}$$

which proved convexity of functionΨ
2.

$$\Psi(y) - \Psi(z) \leq \sum_{i=1}^{k} \left(z_i - y_i\right) \frac{\exp(-y_i)}{\sum_{i=1}^{k} \exp(-y_i)}$$