CONVERGENCE AND UNIQUENESS OF PAGERANK

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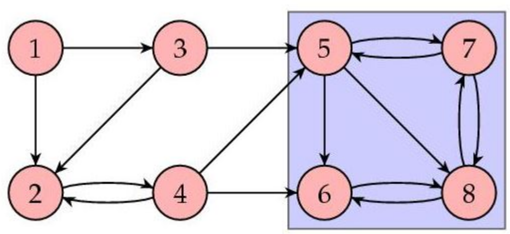
Convergence of Power Method $x^{(k+1)} = P^T x^{(k)}$ by iterations: Here P^T is a column stochastic matrix, where each column sum is 1, and all the entries are non-negative. In order to generate the stochastic matrix in PageRank method, we will consider the adjacent matrix A and the degree diagonal matrix D. This $x^{(k+1)} = P^T x^{(k)}$ method is same with searching stationary distribution for a Markov chain. To grantee the existence of the unique stationary distribution vector for the Markov equation, the transition matrix P is **irreducible** and **primitive**.

A square nxn matrix A is called **reducible** if the indices $1, 2, \ldots n$ can be divided into two disjoint nonempty sets $i_1, i_2, \ldots i_\mu$ and $j_1, j_2, \ldots, j_v$, where $\mu + v = n$ such that $a_{i_\alpha j_\beta} = 0$ for $\alpha = 1, 2, \ldots, \mu$ and $\beta = 1, 2, \ldots, v$.

If the matrix P is irreducible, there is a graph inside of the big network. This graph may become a absorbing class as see in the Figure 1.



The stochastic matrix P for the Figure 1 is as following:

$$P^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

the stationary vector is

$$x^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.12 & 0.24 & 0.24 & 0.4 \end{bmatrix}$$

All the PageRank run into states in the small graph.

A **Primitive** matrix is a square non-negative matrix some power of which is positive.

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Perron-Frobenius Theorem for Primitive Non-negative Matrices
Let$A = A_{ij}$ be an $n \times n$ non-negative primitive matrix, and let $T(v)$ be the associated lineaer map defined by

$$T(v)_j = \sum_{i=1}^{n} v_i A_{ij} \text{for each} i, j. \tag{1}$$

Then,

- T has a postive simple eigenvalue λ, and unique positive eigenvector x with $abs(x) = 1$ such that all eigenvectors corresponding to λ and non-zero multiples of x.
- Any eigenvalue μ of T with $\mu \neq \lambda$ is such that $|\mu| < \lambda$.

Apply the Perron-Frobenius theorem on P (a stochastic matrix with row sum of 1) , there is only one eigenvalue on its spectral circle. This also applied on the P^T.
The eigenvalues of P^T are λ_i, and

$$1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{2}$$

Hence the power method applied to a primitive stochastic matrix P, the stationary eigenvector always approach to the eigenvector responses to the largest eigenvalue 1. This is guaranteed to converge to the unique dominant eigenvector,which is the PageRank for each node/page.

References

[1] A.N. and Meyer Langville C.D., *Deeper inside PageRank*, Internet Mathematics **1** (2004), no. 3, 335–380.

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