MATH 221, Fall 2016 - Homework 1 Solutions

Due Tuesday, September 13

Section 1.1

Page 10, Problem 1:

The linear system $\begin{array}{r} x_1 + 5x_2 = 7 \\ -2x_1 - 7x_2 = -5 \end{array}$ in matrix form is $\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$. Using row operations yields:

$2R_1 + R_2 \to R_2$ **:** $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$ $\frac{1}{3}R_2 \to R_2$ **:** $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$ $-5R_2 + R_1 \to R_1$ **:** $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$

The final matrix, $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$, is equivalent to the system $\begin{array}{r} x_1 = -8 \\ x_2 = 3 \end{array}$, which is the ordered pair $(-8, 3)$**.**

Page 10, Problem 3:

To find the point of intersection of the lines amounts to solving the linear system $\begin{array}{r} x_1 + 2x_2 = 4 \\ x_1 - x_2 = 1 \end{array}$.

Using row operations on the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ gives $-R_1 + R_2 \to R_2$ **:** $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \end{bmatrix}$ $-\frac{1}{3}R_2 \to R_2$ **:** $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

$-2R_2 + R_1 \to R_1$ **:** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, which is equivalent to the system $\begin{array}{r} x_1 = 2 \\ x_2 = 1 \end{array}$. The point of intersection is therefore $(2, 1)$.

Page 10, Problem 12:

The system $\begin{array}{r} x_1 - 5x_2 + 4x_3 = -3 \\ 2x_1 - 7x_2 + 3x_3 = -2 \\ -2x_1 + x_2 + 7x_3 = -1 \end{array}$ in matrix form is $\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix}$. Using row operations yields:

$R_2 + R_3 \to R_3$**:** $\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ 0 & -6 & 10 & -3 \end{bmatrix}$ $-2R_1 + R_2 \to R_2$ **:** $\begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -6 & 10 & -3 \end{bmatrix}$ $2R_2 + R_3 \to R_3$**:** $\begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

At this point, the last matrix is equivalent to the system $\begin{array}{r} x_1 - 5x_2 + 4x_3 = -3 \\ 3x_2 - 5x_3 = 4 \\ 0 = 5 \end{array}$.

The third equation, $0 = 5$, is never true, so there is no solution. Therefore, the system is inconsistent.

Page 10, Problem 17:

If the lines have a common point of intersection, then the solution to the system $\begin{array}{c} 2x_1 + 3x_2 = -1 \\ 6x_1 + 5x_2 = 0 \\ 2x_1 - 5x_2 = 7 \end{array}$ should be consistent.

To begin solving this system, interchange rows 2 and 3 of the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{bmatrix}$ so it becomes $\begin{bmatrix} 2 & 3 & -1 \\ 2 & -5 & 7 \\ 6 & 5 & 0 \end{bmatrix}$.

Perform row operations on this matrix: $-R_1 + R_2 \to R_2$: $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -8 & 8 \\ 6 & 5 & 0 \end{bmatrix}$ $-3R_1 + R_3 \to R_3$: $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -8 & 8 \\ 0 & -4 & 3 \end{bmatrix}$

$-2R_3 + R_2 \to R_3$: $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -8 & 8 \\ 0 & 0 & 2 \end{bmatrix}$. This matrix is equivalent to the system $\begin{array}{c} 2x_1 + 3x_2 = -1 \\ -8x_2 = 8 \\ 0 = 2 \end{array}$, which is inconsistent.

Therefore, there is no solution and the lines do not have a common point of intersection.

Page 10, Problem 21:

Recall that a system is consistent if there exists one solution or infinitely many solutions. Also recall that a pivot is a non-zero number in a matrix that has only zeros to the left of it in its row when the matrix is in row-echelon form.

A system is consistent if its augmented matrix has no pivots in the last column (which would yield no solution).

Therefore, it is necessary to perform row operations on the matrix in order to "solve the system".

Row reduction on the matrix $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$ yields: $-3R_1 + R_2 \to R_2$ **:** $\begin{bmatrix} 1 & 4 & -2 \\ 0 & -12 + h & 0 \end{bmatrix}$.

Using arbitrary variables x_1 and x_2, this matrix is equivalent to the system: $\begin{array}{c} x_1 + 4x_2 = -2 \\ (-12 + h)x_2 = 0 \end{array}$. If h = 12, then the system has infinite solutions. If $h \neq 12$, then $x_2 = 0$. Either way, the system is consistent. So, the soluton is all h.

Page 11, Problem 23a:

True or False: Every elementary row operation is reversible. **TRUE**

From the text on page 6: "It is important to note that row operations are reversible"

In performing row operations, you transform one matrix into another, so it must be possible to transform the new matrix into the old by reverse operations.

Section 1.2

Page 22, Problem 10:

In order to find the general solution, the matrix needs to be in reduced echelon form. To do so, use row operations:

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix} 2R_1 + R_2 \to R_2 : \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{bmatrix} -\tfrac{1}{7}R_2 \to R_2\text{:} \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_2 + R_1 \to R_1$: $\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. This is equivalent to the system $\begin{matrix} x_1 - 2x_2 = 2 \\ x_3 = -2 \end{matrix}$. Therefore, the solution is:

$$\begin{cases} x_1 = 2 + 2x_2 \\ x_2 \text{ is free} \\ x_3 = -2 \end{cases}$$

Page 22, Problem 12:

The system corresponding to the matrix (already in reduced echelon form) $\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{matrix} x_1 - 9x_3 = 4 \\ x_2 + 3x_3 = -1 \\ x_4 = -7 \\ 0 = 1 \end{matrix}$

This system is obviously inconsistent because there is a pivot in the last column of the matrix, so no solution exists.

Page 22, Problem 14:

In order to find the general solution, the matrix must be in reduced echelon form. However, the matrix

$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is **not** in reduced echelon form (because the -8 is in the same column as a pivot).

Row reducition results in: $8R_3 + R_1 \to R_1$: $\begin{bmatrix} 1 & 0 & -5 & 0 & 0 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which is the system: $\begin{matrix} x_1 - 5x_3 = 3 \\ x_2 + 4x_3 - x_4 = 6 \\ x_5 = 0 \end{matrix}$.

Solving for the basic variables in terms of the free variables gives the general solution: $\begin{cases} x_1 = 3 + 5x_3 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

Page 22, Problem 20:

For each case in this problem, it is necessary to rewrite the system as an augmented matrix in its echelon form:

$$\begin{matrix} x_1 - 3x_2 = 1 \\ 2x_1 + hx_2 = k \end{matrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} -2R_1 + R_2 \to R_2 : \begin{bmatrix} 1 & -3 & 1 \\ 0 & 6+h & -2+k \end{bmatrix}$$

(a) As previously stated, a system is inconsistent (no solution) if there is a pivot in the last column of the augmented matrix. Therefore, $6 + h = 0$ in order for the last column to be a pivot and $-2 + k \neq 0$. So, $h = -6$ and $k \neq 2$.

Page 22, Problem 20 (cont):

(b) A system has a unique solution when there is no pivot in the last column and there are no free variables.

Therefore, $6+h \neq 0$ and any value of k is acceptable. A unique solution exists for any k as long as $h \neq -6$.

(c) A system has many solutions when there exists a free variable. In this case, that is when the bottom row is all 0.

This happens when $6+h=0$ and $-2+k=0$ resulting in the solution $h=-6$ and $k=2$.

Page 23, Problem 31:

Yes, an overdetermined system can be consistent (it can have a unique solution or infinitely many solutions).

Specific examples will vary. It is best to think of an augmented matrix in reduced echelon form that describes a system:

- $\begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 0 \end{bmatrix}$ where A and B are constants. This would be an example of a consistent system with a unique solution. In this case, one of the equations is a multiple of the others. To find a system, you can perform row operations to yield a unique set of equations such as $R_2 + R_1 \rightarrow R_1$: $\begin{bmatrix} 1 & 1 & A+B \\ 0 & 1 & B \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 + R_2 \rightarrow R_2$: $\begin{bmatrix} 1 & 1 & A+B \\ 1 & 2 & A+2B \\ 0 & 0 & 0 \end{bmatrix}$ $CR_2 + R_3 \rightarrow R_3$: $\begin{bmatrix} 1 & 1 & A+B \\ 1 & 2 & A+2B \\ C & 2C & AC+2BC \end{bmatrix}$, where C is a constant (not 0). For example, let A = 1, B = 2, and C = 3, which will result in the matrix: $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & 6 & 15 \end{bmatrix}$, equivalent to the system $\begin{matrix} x_1 + x_2 = 3 \\ x_1 + 2x_2 = 5 \\ 3x_1 + 6x_2 = 15 \end{matrix}$, the solution of which is $\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$

- $\begin{bmatrix} 1 & 1 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where A is a constant. This would be an example of a consistent system with infinitely many solutions. In this case, two of the equations are multiples of the first equation. Perform row operations: $BR_1 + R_2 \rightarrow R_2$, $CR_1 + R_3 \rightarrow R_3$: $\begin{bmatrix} 1 & 1 & A \\ B & B & AB \\ C & C & AC \end{bmatrix}$, where B and C are non-zero constants. For example, let A = 1, B = 2, and C = 3, which will result in the matrix: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, equivalent to the system $\begin{matrix} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_3 + 3x_3 = 3 \end{matrix}$, the solution of which is $\begin{cases} x_1 = 1 - x_2 \\ x_2 \text{ is free} \end{cases}$

Section 1.3

Page 32, Problem 9:

From the system $\begin{matrix} x_2 + 5x_3 = 0 \\ 4x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 3x_2 - 8x_3 = 0 \end{matrix}$, let the scalars be x_1, x_2, and x_3. The vectors are the coefficients of the

scalars $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix}$ and the constant terms on the right side of the equation

Page 32, Problem 9 (cont):

$\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, the linear system is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{c}$.

Page 32, Problem 10:

Using the same method from Problem 9 in this section, $\begin{array}{c} 3x_1 - 2x_2 + 4x_3 = 3 \\ -2x_1 - 7x_2 + 5x_3 = 1 \\ 5x_1 + 4x_2 - 3x_3 = 2 \end{array}$ is equivalent to the vector equation

with scalars x_1, x_2, and x_3 and vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$:

$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{c}$.

Page 32, Problem 12:

Asking whether the vectors form a linear combination of vector **b** is equiavelnt to determining if the linear system

that forms the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution. The matrix is $\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$ and row

operations result in: $-R_1 + R_3 \to R_3$: $\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & 2 \end{bmatrix}$. Because there is a pivot in every row and none in the

right-most column, the linear system is consistent, and hence the vectors **do** form a linear combination of the vector **b**.

Page 32, Problem 14:

Asking whether the vectors formed by the columns of matrix A form a linear combination of vector **b** is equiavelnt

to determining if the linear system that forms the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution. Therefore, the

matrix is $\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$, and performing row operations $2R_1 + R_2 \to R_2$: $\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix}$ $-2R_2 + R_3 \to R_3$:

$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is a consistent system (with infinitely many solutions). Therefore, the vectors of the columns of

the matrix A **do** form a linear combination of the vector **b.**

Page 32, Problem 16:

This question is asking for what value of h is **y** in *Span* $\{\mathbf{v}_1, \mathbf{v}_2\}$. In $\mathbb{R}^3$, the span of two nonzero vectors (with neither the multiple of the other) is a plane that contains the two vectors and the origin in addition to all the vectors that can be written as a linear combination of the two vectors. Therefore, to determine when **y** is in this plane, determine when the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{y}$ is consistent. To do so, write the system as a linear combination and reduce:

Page 32, Problem 16 (cont):

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} 2R_1 + R_3 \to R_3 : \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h + -5 \end{bmatrix} -3R_2 + R_3 \to R_3 : \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h + 4 \end{bmatrix}.$$ The system will only be consistent when there is no pivot in the right-most column, so $2h + 4 = 0$ in order for there to be no pivot in that position. So, $h = -2$.

Page 32, Problem 23c:

True or False: An example of a linear combination of the vectors $\mathbf{v}_1$and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$. **TRUE**

Consider the linear combination $\frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2$ (on page 28 of the text).

Page 32, Problem 23d:

True or False: The solution set of the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$. **TRUE**

This is defined in the box on page 29 of the text.

Page 32, Problem 23e:

True or False: The set $Span\{\mathbf{u}, \mathbf{v}\}$is always visualized as a plane through the origin. **FALSE**

This is true only when **u** and **v** are both nonzero with **v** not a multiple of **u** (as explained on page 30 in the text).