MATH 221, Fall 2016 - Homework 2 Solutions

Due Tuesday, September 20

Section 1.4

Page 40, Problem 2:

The product is **not defined** because the order of the matrix is 3x1 and the order of the vector is 2x1. The number of columns of the first matrix (3) does not equal the number of entries of the vector (2).

Page 40, Problem 4:

The product **is defined** because the order of the matrix is 2x3 and the vector is 3x1 (so the number of columns (3) in the matrix is equal to the number of entries in the vector). The order of the product should be 2x1, the number of rows of the matrix and the number of entries of the vector.

a. Using the definition, as in Example 1 on page 35:

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

b. Using the row-vector rule (explained on page 38):

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 3(2) + -4(1) \\ 3(1) + 2(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Page 40, Problem 6:

This exercise is similar to part a of the problem 4, which is like Example 1. Use the elements of the vector as scalars for the columns of the matrix:

$$-3 \cdot \begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5 \cdot \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

Page 40, Problem 8:

This is similar to the previous exercise, but now write the column vectors as a 2x4 matrix, the scalars as a 4x1 column-vector, and keep the left-side of the equation as a two-column vector:

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Page 40 Problem 9:

Vector Equation: $x_1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ Matrix Equation: $\begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

Page 40, Problem 12:

Augmented Matrix: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix}$ Row-Reduction: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ The solution, as a vector: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$

Page 40, Problem 13:

To answer this question, determine if $\mathbf{u}$ is a linear combination of the columns of A. That is, determine if

$A\mathbf{x} = \mathbf{u}$ has a solution. The augmented matrix is $\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ and row-reduction yields:

$\begin{bmatrix} 1 & 1 & 4 \\ 3 & -5 & 0 \\ -2 & 6 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -8 & -12 \\ 0 & 8 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Because there is no pivot in the last column, a

solution exists, so **u is in the plane in $\mathbb{R}^3$ spanned by the columns of A**.

Page 40, Problem 14:

This question is answered in the same way as above. That is, determine if $A\mathbf{x} = \mathbf{u}$ has a solution.

The augmented matrix is $\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix}$ and row-reduction yields:

$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. Because there is a pivot in the

last column, no solution exists, so **u is NOT in the subset of $\mathbb{R}^3$ spanned by the columns of A**.

Page 41, Problem 22:

The matrix formed by these vectors is $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} -3 & 9 & -6 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

It is clear that there is a pivot in each row, so the vectors span $\mathbb{R}^3$ by Theorem 4 of this section.

Page 42, Problem 34:

We know $\mathbf{v}_1 = A\mathbf{u}_1$ and $\mathbf{v}_2 = A\mathbf{u}_2$ are consistent and $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. So, $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 = A\mathbf{u}_1 + A\mathbf{u}_2$. By

Theorem 5a of this section, $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2 = A(\mathbf{u}_1 + \mathbf{u}_2)$. Therefore, $\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2$ is a solution to $A\mathbf{x} = \mathbf{w}$.

Page 42, Problem 35:

Assume $\mathrm{A}\mathbf{y} = \mathbf{z}$ is true. Then, $5\mathbf{z} = 5\mathrm{A}\mathbf{y} = \mathrm{A}(5\mathbf{y})$ (by Theorem 5b on page 39). Let $\mathbf{x} = 5\mathbf{y}$. Then, $\mathrm{A}\mathbf{x} = 5\mathbf{z}$ is also consistent.

Section 1.5

Page 47, Problem 2:

Use row operations on the augmented matrix: $\begin{bmatrix} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & -7 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

Because there is a pivot in every column of the coefficient matrix, there are no free variables, **so the system has only the trivial solution**.

Page 47, Problem 13:

As vectors, this line is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$, which is a line through $\begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$.

Page 47, Problem 15:

First, realize that the second equation is the first equation shifted by 2. Solving the first equation for x_1 results in $x_1 = -5x_2 + 3x_3$. In vector form, this is the same as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, which is a plane through the origin spanned by $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. The solution to the second equation is:

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, which is a parallel plane through $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ instead of $\mathbf{0}$.

Page 48, Problem 38:

By Theorem 5b on page 39, $A(c\mathbf{w}) = cA\mathbf{w}$. Since $\mathbf{w}$ satisfies $\mathrm{A}\mathbf{x} = \mathbf{0}$, $\mathrm{A}\mathbf{w} = \mathbf{0}$. So, $cA\mathbf{w} = c\mathbf{0} = \mathbf{0}$, so $A(c\mathbf{w}) = \mathbf{0}$.