MATH 221, Fall 2016 - Homework 3 Solutions

Due Tuesday, September 27

Section 1.5

Page 47, Problem 8:

In order to solve this problem, put the matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0} \end{bmatrix}$ (where $\mathbf{a}_1$, etc. are the columns of A)

in reduced echelon form: $\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$, which is equivalent to the

system $\begin{matrix} x_1 - 2x_3 - 7x_4 = 0 \\ x_2 + 2x_3 - 4x_4 = 0 \end{matrix}$. It is clear that the basic variables are x_1 and x_2 while the free varaibles are x_3

and x_4. Solving for the free variables results in: $\begin{matrix} x_1 = 2x_3 + 7x_4 \\ x_2 = -2x_3 + 4x_4 \end{matrix}$. Writing in parametric vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Page 47, Problem 10:

This is the same process as problem 8 in this section: $\begin{bmatrix} -1 & -4 & 0 & -4 & 0 \\ 2 & -8 & 0 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{matrix} x_1 = -4x_4 \\ x_2 = 0 \end{matrix}$. The basic variables are x_1 and x_2 while the free variables are x_3 and x_4. The parametric vector

form is: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Page 47, Problem 12:

This is the same process as the previous two problems: $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1 = 2x_2 - 3x_3 - 29x_5 \\ x_4 = -4x_5 \\ x_6 = 0 \end{matrix}$. The basic variables are x_1, x_4, and x_6.

The free variables are x_2, x_3, and x_5. The solution in parametric vector form is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Page 47, Problem 18:

The system as an augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix}$ and row reduction yields: $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the parametric solution being $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$.

This solution is a line through $\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution to the homogenous equation in Exercise 6.

Page 48, Problem 35:

By inspection, the second column of A, $\mathbf{a}_2 = 3\mathbf{a}_1$. Therefore, one **nontrivial** (not $\mathbf{0}$) solution is

$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ or $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Section 1.7

Page 60, Problem 6:

Determine if A$\mathbf{x} = \mathbf{0}$ has only the trivial solution:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ -4 & -3 & 0 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & -20 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because there are no free variables, the system has only the trivial solution, so **the columns of A form a linearly independent set.**

Page 60, Problem 8:

You could use the same process as above, but notice that there are 4 vectors in $\mathbb{R}^3$ (because the matrix is 3 x 4).

By Theorem 8 of this section, **the vectors are linearly dependent** (there must be at least one free variable, if a solution exists).

Page 61, Problem 14:

In order for the vectors to be linearly dependent, the system A$\mathbf{x} = \mathbf{0}$ (where A is a matrix formed by the column vectors) must have a nontrivial solution.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 1 & 0 \\ -4 & 6 & h & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -6 & 8+h & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 38+h & 0 \end{bmatrix}.$$

A nontrivial solution exists when there is a free variable. Therefore, a nontrivial solution exists for $h = -38$.

Page 61, Problem 18:

This a set of 4 vectors in $\mathbb{R}^2$. By Theorem 8, because p = 4 > 2 = n, the **set of vectors is linearly dependent.**

Page 61, Problem 20:

By Theorem 9, any set that contains the zero vector is linearly dependent. Thus, **this set is linearly dependent.**

Page 61, Problem 21a:

True or False: The columns of a matrix A are linearly independent if the equation A$\mathbf{x} = \mathbf{0}$ has the trivial solution.

FALSE - A homogenous system always has the trvial solution (as explained on page 56). The question of linear independence is whether the trivial solution is the **only** solution.

Page 61, Problem 21b:

True or False: If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

FALSE - Not all vectors need to be linear combinations of each other. At least one of the vectors needs to be a linear combination of the others (see Thereom 7 and the following warning on page 58).

Page 61, Problem 21c:

True or False: The columns of any 4 x 5 matrix are linearly dependent.

TRUE - In this case, there are 5 vectors in $\mathbb{R}^4$. By Theorem 8 in this section, because n = 4 < 5 = p, the set of vectors formed by the columns of this matrix are linearly dependent.

Page 61, Problem 21d:

If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in $Span\,\{\mathbf{x}, \mathbf{y}\}$.

TRUE - Because $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent but $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, $\mathbf{z}$ must be a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Thus, $\mathbf{z}$ must be in $Span\,\{\mathbf{x}, \mathbf{y}\}$.

Page 61, Problem 33:

TRUE - Because $\mathbf{v}_3$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, the set is linearly dependent, by Theorem 7 of this section.