MATH 221, Fall 2016 - Homework 5 Solutions

Due Tuesday, October 11

Section 2.1

Page 100, Problem 3:

To begin, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -3 & 5 \end{bmatrix}$ and

$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -15 \\ 9 & -6 \end{bmatrix}$

Page 100, Problem 5:

a. $A\mathbf{b}_1 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 26 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ -19 \end{bmatrix}$ So, $AB = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1(4)+3(-2) & -1(-2)+3(3) \\ 2(4)+4(-2) & 2(-2)+4(3) \\ 5(4)+-3(-2) & 5(-2)+-3(3) \end{bmatrix} = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

Page 100, Problem 6:

a. $A\mathbf{b}_1 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 12 \\ 3 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 22 \\ -22 \\ -2 \end{bmatrix}$ So, $AB = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

b. $AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4(1)+-3(3) & 4(4)+-3(-2) \\ -3(1)+5(3) & -3(4)+5(-2) \\ 0(1)+1(3) & 0(4)+1(-2) \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

Page 100, Problem 12:

Because A is 2x2 and B is 2x2, our new matrix of all zeros will also be 2x2. Essentially, we want to solve

$\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ with non-zero columns. Multiplying these matrices results in a linear system:

$\begin{aligned} 3a - 6c &= 0 \\ 3b - 6d &= 0 \\ -2a + 4c &= 0 \\ -2b + 4d &= 0 \end{aligned}$, which can be broken into two separate systems: $\begin{aligned} 3a - 6c &= 0 \\ -2a + 4c &= 0 \end{aligned}$ and $\begin{aligned} 3b - 6d &= 0 \\ -2b + 4d &= 0 \end{aligned}$.

Using row reduction, $\begin{bmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $a = 2c$ and $b = 2d$. Answers will vary.

An example is $c = 1$, $d = 1$ so $a = b = 2$: $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Page 101, Problem 24:

Remember, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $D = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix}$. By definition of matrix multiplication, the columns of AD are equivalent to $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$, respectively. In order for $AD = I_3$, the systems generated by $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$ must each have at least one solution. Since the columns of A span $\mathbb{R}^3$, each of theses systems do have at least one solution (see Theorem 4 in Section 1.4). So, the matrix D is found by selecting one of the solutions from each of the systems ($A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$) and using it as the columns of D.

Page 101, Problem 26:

Let $\mathbf{b} \in \mathbb{R}^{\mathrm{m}}$ be arbitrary ($\mathbf{b}$ is an m x 1 matrix or vector). Assume $AD = I_m$ is true. Then, multiplying by $\mathbf{b}$ yields $AD\mathbf{b} = I_m\mathbf{b}$, which implies $AD\mathbf{b} = \mathbf{b}$ (because $I_m$is essentially 1). Because the order of the matrices is defined, $A(D\mathbf{b}) = \mathbf{b}$ (by Theorem 2 of this section on page 97). The product $D\mathbf{b}$ is a vector which can be written as $\mathbf{x} = D\mathbf{b}$. So, $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$. By Theorem 4 in Section 1.4, since $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$, A has a pivot position in every row. Because each pivot is in a different column, A must have at least as many columns as rows.

Page 101, Problem 33:

Let A be an arbitrary matrix of order i x j: $A = \begin{bmatrix} a_{11} & \dots & a_{1j} \\ \vdots & \dots & \vdots \\ a_{i1} & \cdots & a_{ij} \end{bmatrix}$ and B of order j x k : $B = \begin{bmatrix} b_{11} & \dots & b_{1k} \\ \vdots & \dots & \vdots \\ b_{j1} & \cdots & b_{jk} \end{bmatrix}$.

The product AB is defined because the number of columns of A (j) equals the number of rows of B (j).

The product is: $AB = \begin{bmatrix} a_{11}b_{11} + \ldots + a_{1j}b_{j1} & \cdots & a_{11}b_{1k} + \ldots + a_{1j}b_{jk} \\ \vdots & \dots & \vdots \\ a_{i1}b_{11} + ... + a_{ij}b_{j1} & \cdots & a_{i1}b_{1k} + \ldots + a_{ij}b_{jk} \end{bmatrix}$ which is a matrix of order i x k.

It follows that $(AB)^T = \begin{bmatrix} a_{11}b_{11} + \ldots + a_{1j}b_{j1} & \cdots & a_{i1}b_{11} + ... + a_{ij}b_{j1} \\ \vdots & \dots & \vdots \\ a_{11}b_{1k} + \ldots + a_{1j}b_{jk} & \cdots & a_{i1}b_{1k} + \ldots + a_{ij}b_{jk} \end{bmatrix}$, which is a matrix of order k x i.

$B^T = \begin{bmatrix} b_{11} & \dots & b_{j1} \\ \vdots & \dots & \vdots \\ b_{1k} & \cdots & b_{jk} \end{bmatrix}$, which is of order k x j, and $A^T = \begin{bmatrix} a_{11} & \dots & a_{i1} \\ \vdots & \dots & \vdots \\ a_{1j} & \cdots & a_{ij} \end{bmatrix}$, which is of order j x i.

The product B^TA^T is: $\begin{bmatrix} b_{11}a_{11} + \ldots + b_{j1}a_{1j} & \dots & b_{11}a_{i1} + \ldots + b_{j1}a_{ij} \\ \vdots & \dots & \vdots \\ b_{1k}a_{11} + \ldots + b_{jk}a_{1j} & \cdots & b_{1k}a_{i1} + \ldots + b_{jk}a_{ij} \end{bmatrix}$, which is a matrix of order k x i.

Looking at $(AB)^T$ and B^TA^T, it is clear that the matrices are equivalent.

Section 2.2

Page 109, Problem 4:

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \; A^{-1} = \frac{1}{-12+16} \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

Page 109, Problem 9a:

True or False: In order for a matrix B to be the inverse of A, the equations AB = I and BA = I must both be true.

TRUE - This is the definition of invertible on page 103.

Page 109, Problem 9b:

True or False: If A and B are n x n and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

FALSE - By Theorem 6 on page 105, $(AB)^{-1} = B^{-1}A^{-1}$, which does not always equal $A^{-1}B^{-1}$.

Page 109, Problem 9c:

True or False: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

FALSE - By Theorem 4 of this section, a 2 x 2 matrix is invertible if and only if $ad - bc \neq 0$.

The expression $ab - cd$ reveals nothing about the invertibility of a matrix.

For example, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow ab - cd = 1 - 0 \neq 0$, but the matrix is not invertible because $ad - bc = 0$.

Page 109, Problem 9d:

True or False: If A is an invertible n x n matrix, then the equation A**x** = **b** is consistent for each **b** in $\mathbb{R}^n$.

TRUE - This follows from Theorem 5 of this section on page 104.

Page 110, Problem 14:

Because $(B - C)$ is an m x n matrix, D must be an n x n matrix (because the product $(B - C)D$ is defined

and D is invertible). Thus, 0 is an m x n matrix. Beacuse D is invertible,

$(B - C)DD^{-1} = 0 \cdot D^{-1} \Rightarrow (B - C)I_n = 0$, where 0 is still an m x n matrix because D^{-1} is still n x n.

Thus, $B - C = 0$ because I_n is essentially 1. Thus, $B - C + C = 0 + C \Rightarrow B + (-C + C) = 0 + C \Rightarrow B = C$.

Page 110, Problem 16:

Because A and B are both n x n matrices, their products and inverses (if they exist) are also n x n.

Using the hint, let $C = AB$ and solve for A: $CB^{-1} = ABB^{-1} \Rightarrow CB^{-1} = A$, but $C = AB$.

Therefore, A is the product of invertible matrices. By Theorem 6 of this section, A must also be invertible.

Page 110, Problem 18:

Because the order of all matrices is n x n, their products and inverses (if they exist) are also n x n.

Because B is invertible, $ABB^{-1} = BCB^{-1} \Rightarrow AI_n = BCB^{-1} \Rightarrow A = BCB^{-1}$.

Page 110, Problem 31:

To find the inverse, use the algorithm on page 108:

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

So, the inverse is $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.