MATH 221, Fall 2016 - Homework 7 Solutions

Due Tuesday, October 25

Section 4.1

Page 196, Problem 16:

It is clear that W is not a vector space because it can never contain the zero vector (the first entry is always 1).

Page 196, Problem 21:

The set H is a subspace of M_{2x2} because:

1) If $a = b = d = 0$, the zero vector is contained in the space.

Let $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ be two arbitrary matrices in H.

2) Then, $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, so H is closed under addition.

3) Let β be an arbitrary scalar. Then, $\beta \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} \beta a_1 & \beta b_1 \\ 0 & \beta d_1 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

So H is closed under scalar multiplication.

Page 196, Problem 22:

The set M_{2x4} is the set of all matrices of the form $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ where the entries are arbitrary.

This set is a subspace (as stated in the problem).

Let the matrix F be $F = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix}$ where the entries are fixed.

The set $H = \{A \in M_{2x4} : FA = 0\}$ is a subset of M_{2x4}. To show H is a subspace:

1) Because $F0 = 0$, $0 \in H$.

2) Let A_1 and A_2 be arbitrary matrices in H. Then, $F(A_1) = 0$ and $F(A_2) = 0$.

Because$F(A_1 + A_2) = F(A_1) + F(A_2) = 0 + 0 = 0$. Thus, $A_1 + A_2 \in H$, so H is closed under addition.

3) Let $A \in H$ and $c \in \mathbb{R}$ be arbitrary. Thus, $FA = 0$. So, $F(cA) = cFA = c(FA) = 0$.

Thus, $cA \in H$, so H is closed under scalar multiplication.

Page 197, Problem 32:

To show $H \cap K$ is a subspace, check the three conditions:

1) Because H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Thus, $\mathbf{0} \in H \cap K$.

2) Let $\mathbf{u} \in H \cap K$ and $\mathbf{v} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$ and $\mathbf{v} \in H$ and $\mathbf{v} \in K$.

Because H and K are subspaces, $\mathbf{u} + \mathbf{v} \in H$ and $\mathbf{u} + \mathbf{v} \in K$. Thus, $\mathbf{u} + \mathbf{v} \in H \cap K$.

3) Let $c \in \mathbb{R}$ and $\mathbf{u} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$.

Because H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{u} \in K$. Thus, $c\mathbf{u} \in H \cap K$.

An example in $\mathbb{R}^2$ to show $H \cup K$ is not always a subspace would be $H = \{(x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, y) : y \in \mathbb{R}\}$

(the x-axis and y-axis, respectively). Let $\mathbf{u} =(1, 0) \in H \cup K$ and $\mathbf{v} = (0, 1) \in H \cup K$.

Then, $\mathbf{u} + \mathbf{v} = (1, 1)$, which is not in H or in K, so it is not in $H \cup K$.

Thus, $H \cup K$ is not closed under addition and is therefore not a subspace.

Page 197, Problem 33a:

To show $H + K$ is a subspace:

1) Because H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Thus, $\mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{0} \in H + K$.

2) Let $\mathbf{x}, \mathbf{y} \in H + K$ be arbitrary.

Then, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in H$ and $\mathbf{v} \in K$ and $\mathbf{y} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s} \in H$ and $\mathbf{t} \in K$.

Thus, $\mathbf{x} + \mathbf{y} = (\mathbf{u} + \mathbf{v}) + (\mathbf{s} + \mathbf{t}) = (\mathbf{u} + \mathbf{s}) + (\mathbf{v} + \mathbf{t})$. But, $\mathbf{u} + \mathbf{s} \in H$ and $\mathbf{v} + \mathbf{t} \in K$ (because H and K are subspaces).

Thus, $\mathbf{x} + \mathbf{y} \in H + K$, so $H + K$ is closed under addition.

3) Let $\mathbf{x} \in H + K$ and $c \in \mathbb{R}$ be arbitrary. Then, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in H$ and $\mathbf{v} \in K$. Thus, $c\mathbf{x} = c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Because H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$. Thus, $c\mathbf{x} \in H + K$, so $H + K$ is closed under scalar multiplication.

Page 197, Problem 33b:

Because every vector in H can be written as a sum of itself and $\mathbf{0}$ (the zero vector in K and H), H is a subset of $H + K$.

Because H contains the zero vector and H is closed under addition and scalar multiplication (because H is a subspace of V), H is a subspace of $H + K$ (this arguement also applies to K, so K is also a subspace of $H + K$.

Section 4.2

Page 206, Problem 6:

Solve the equation $A\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So, a spanning set for the null space is $\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 206, Problem 9:

The system of equations can be rearranged to $\begin{array}{c} p - 3q - 4s - 0r = 0 \\ 2p - 0q - s - 5r = 0 \end{array}$. So the vectors in W are solutions to this system.

Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 2 (and hence a vector space).

Page 206, Problem 14:

Notice that $W = \mathit{Col}\,\mathrm{A}$ for $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix}$. Therefore, W is a subspce of $\mathbb{R}^3$ (and a vector space) by Theorem 3

(look at Example 4 of this section).

Page 206, Problem 27:

Let $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$. Then, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} \in \mathit{Nul}\,\mathrm{A}$. Since $\mathit{Nul}\,\mathrm{A}$ is a subspace

of $\mathbb{R}^3$, it is closed under scalar multiplication. Therefore, $10\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also in $\mathit{Nul}\,\mathrm{A}$ (a solution to the system).

Page 206, Problem 28:

Let $A = \begin{bmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$. Because there is a solution to $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \in \mathit{Col}\,\mathrm{A}$. Since $\mathit{Col}\,\mathrm{A}$ is a subspace

of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus, $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in $\mathit{Col}\,\mathrm{A}$. So, the second system must also have

a solution.

Page 207, Problem 30:

Let $T(\mathbf{x})$ and $T(\mathbf{w})$ be vectors in the range of T. Then, because T is a linear transformation, $T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w})$ and for any scalar c, $cT(\mathbf{x}) = T(c\mathbf{x})$. Because $T(\mathbf{x} + \mathbf{w})$ and $T(c\mathbf{x})$ are in the range of T (which is a subset of W), it follows that the range of T is a subspace of W (closed under addition and scalar multiplication).