MATH 221, Fall 2016 - Homework 8 Solutions

Due Tuesday, November 8

Section 4.3

Page 213, Problem 3:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$,which reduces to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there are only two pivot positions, **the set of vectors are neither linearly independent nor span** $\mathbb{R}^3$, thus **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

Page 213, Problem 8:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix}$. Because there are four columns, **the set cannot be linearly indpendent in** $\mathbb{R}^3$. Thus, **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

To determine if the set of vectors span $\mathbb{R}^3$, row-reduce the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because there is a pivot position in each row, **the set of vectors do span** $\mathbb{R}^3$**.**

Page 213, Problem 13:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1 and 2 (look at matrix B, which is in row echelon form). Use these columns from matrix A to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$. To find a basis for Nul$A$, write the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables (x_3 and x_4): $\mathbf{x} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis for Nul$A$ is $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 214, Problem 14:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1, 3, and 5 (look at matrix$B$, which is in row echelon form). Use these columns from matrix A to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix} \right\}$. To find a basis for Nul$A$, we need the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables (x_2 and x_4). Because matrix B is only in row echelon form, reduce it to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \ \mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \text{ Thus a basis for Nul}A \text{ is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Page 214, Problem 21b:

True or False: If $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

FALSE: The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent.

Page 214, Problem 21c:

True or False: The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^{\text{n}}$.

TRUE: Because the matrix is invertible, the columns span $\mathbb{R}^{\text{n}}$ and are linearly independent (by the Invertible Matrix Theorem). Hence, the columns form a basis for $\mathbb{R}^{\text{n}}$.

Page 214, Problem 21d:

True or False: A basis is a spanning set that is as large as possible.

FALSE: A basis is a spanning set that is as small possible (read "Two Views of a Basis" on p. 212).

Page 214, Problem 22a:

True or False: A linearly independent set in a subspace H is a basis for H.

FALSE: In order to be a basis, the set must also span H (by definition).

Page 214, Problem 22b:

True or False: If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

TRUE: By the Spanning Set Theorem, removing linearly dependent vectors in S will still result in a spanning set (this new set is a subset of S). Because the new set will eventually only contain linearly independent vectors, the set will be a basis for V.

Page 213, Problem 22e:

True or False: If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col$A$.

FALSE: The pivot columns in B tell which columns in matrix A form the basis for Col$A$ (see the warning after Theorem 6 on page 212).

Page 214, Problem 25:

While it might seem that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a spanning set for H, it is not. Notice that H is a subset of Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, there are vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in H, such as $\mathbf{v}_1$ and $\mathbf{v}_3$ (the second and third elements of these vectors are not equal). Therefore, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span H, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be a basis for H.

Page 215, Problem 33:

The polynomials are linearly independent because neither can be written as a scalar multiple of the other. As polynomials in $\mathbb{P}_3$, they can be written as vectors: $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, which as a matrix that is row-reduced is:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, indicating the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution (hence, the columns are linearly independent).

Section 4.4

Page 222, Problem 3:

Let $\mathcal{B}$=[$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$]. Then, $\mathbf{x} = 1\mathbf{b}_1 + 0\mathbf{b}_2 + -2\mathbf{b}_3 = 1\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} + -2\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$.

Page 222, Problem 7:

In this problem, we are solving the equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for the coordinates c_1, c_2, and c_3. In this problem, this equation is represented by $\begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, which amounts to solving the augmented system $\begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix}$. Row-reducing yields $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

So, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

Page 223, Problem 10:

As stated in this section (on page 219), the matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ is the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^{\mathrm{n}}$. Therefore, $P_{\mathcal{B}} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$.

Page 223, Problem 14:

Any polynomial $a + bt + ct^2$ in $\mathbb{P}_2$ can be written in vector form as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Therefore, the set $\mathcal{B}$ as a set of vectors is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the vector $\mathbf{p}$ is $\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$. Solve the augmented system $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{bmatrix}$.

The solution in reduced-echelon form is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

Page 223, Problem 22:

Let $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & ... & \mathbf{b}_n \end{bmatrix}$ (which is an $n \times n$ matrix because its columns form a basis for $\mathbb{R}^{\mathrm{n}}$). By definition, $\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$ which is a transformation of $[\mathbf{x}]_{\mathcal{B}}$ to $\mathbf{x}$. Because the columns of $P_{\mathcal{B}}$ are linearly independent (they form a basis for $\mathbb{R}^{\mathrm{n}}$), $P_{\mathcal{B}}$ is invertible. Thus, left-side multiplication of $P_{\mathcal{B}}^{-1}$ results in $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, which is a transformation of $\mathbf{x}$ to $[\mathbf{x}]_{\mathcal{B}}$ $(\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}})$. Therefore, take $A = P_{\mathcal{B}}^{-1}$.

Page 222, Problem 26:

Assume $\mathbf{w}$ is a linear combination of $\mathbf{u}_1$, ..., $\mathbf{u}_p$. Then, there exist scalars c_1, ..., c_p so that $\mathbf{w} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p$. Since the coordiante mapping $[\mathbf{w}]_{\mathcal{B}}$ is a linear transformation (Theorem 8), it follows that $[\mathbf{w}]_{\mathcal{B}} = c_1 [\mathbf{u}_1]_{\mathcal{B}} + ... + c_p [\mathbf{u}_p]_{\mathcal{B}}$. So, $[\mathbf{w}]_{\mathcal{B}}$ must be a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}$, ..., $[\mathbf{u}_p]_{\mathcal{B}}$. Since the transformation is one-to-one, the converse must be true.