MATH 221, Fall 2016 - Homework 9 Solutions

Due Thursday, November 17

Section 4.5

Page 229, Problem 3:

Any vector in the subspace can be written as $a\begin{bmatrix}0\\1\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\-1\\1\\2\end{bmatrix}+c\begin{bmatrix}2\\0\\-3\\0\end{bmatrix}$. Thus, $\left\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\-1\\1\\2\end{bmatrix},\begin{bmatrix}2\\0\\-3\\0\end{bmatrix}\right\}$ spans the subspace. To determine if this set is linearly independent, solve the matrix equation $\begin{bmatrix}0&0&2\\1&-1&0\\0&1&-3\\1&2&0\end{bmatrix}\mathbf{x}=\mathbf{0}$.

The matrix reduces to $\begin{bmatrix}1&-1&0\\0&1&-3\\0&0&1\\0&0&0\end{bmatrix}$. Thus, the only solution is the trivial solution, so the columns are linearly independent. Therefore, a basis for the subspace is $\left\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\-1\\1\\2\end{bmatrix},\begin{bmatrix}2\\0\\-3\\0\end{bmatrix}\right\}$. Because there are three vectors in the basis, the dimension of the subspace is 3.

Page 229, Problem 8:

The equation can be rewritten as $a = 3b - c$. Thus, any vector $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ in the subspace can be written as $b\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$. Thus, the set $S=\left\{\begin{bmatrix}3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ spans the subspace. It is clear that the set is linearly independent, but to verify that, reduce the matrix formed by the column vectors

$A=\begin{bmatrix}3&-1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, which shows the only solution to $A\mathbf{x}=\mathbf{0}$ is the trivial solution, so the columns are linearly independent. Thus, S is a basis with dimension 3.

Page 229, Problem 10:

Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 15 \end{bmatrix}$. It is clear that the set of these vectors is linearly dependent because $\mathbf{v}_2 = -2\mathbf{v}_1$ and $\mathbf{v}_3 = -3\mathbf{v}_1$. By the Spanning Set Theorem, the set $\{\mathbf{v}_1\}$ still spans $\mathbb{R}^2$ and because the set is linearly independent, it is also a basis for $\mathbb{R}^2$, so the dimension is 1.

Page 229, Problem 14:

Because there are three free variables, the dimension of Nul$A$ is 3 and because there are four pivot positions, the dimension of Col$A$ is 4.

Page 229, Problem 15:

Because there are two free variables, the dimension of Nul$A$ is 2 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 17:

Because there are no free variables, the dimension of Nul$A$ is 0 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 19a:

True or False: The number of pivot columns of a matrix equals the dimension of its column space.

TRUE: This is stated in the box on page 228 before Example 5.

Page 229, Problem 19d:

True or False: If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Page 229, Problem 20d:

True or False: If $\dim V = n$ and if S spans V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Section 4.6

Page 236, Problem 2:

Because rank$A = \dim(\text{Col}A)$, and since there are 3 pivot positions, rank$A = 3$. Because A is a 4×5 matrix, $\dim(\text{Nul}A) + \text{rank}A = 5$. Thus, $\dim(\text{Nul}A) = 5 - 3 = 2$. The basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$ and the

Page 236, Problem 2 (cont):

basis for Row$A$ is the set of non-zero **rows** of B: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$. To find the basis for Nul$A$,

reduce the matrix B to reduced-echelon form to find the solutions to the trivial equation:

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$ So the basis for

Nul$A$ is: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Page 236, Problem 3:

For the same reasons problem 4, rank$A = 3$ and $\dim(\text{Nul}A) = 3$. The basis for Col$A$ is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$

and the basis for Row$A$ is $\left\{ \begin{bmatrix} 2 \\ 6 \\ -6 \\ 6 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. Reducing B results in

$$\begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which implies } \mathbf{x} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the basis for Nul$A$ is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 237, Problem 7:

Because A is a 4×7 matrix, Col$A$ must be a subspace of $\mathbb{R}^4$. Since there are 4 pivot positions, it must be that Col$A = \mathbb{R}^4$.

Nul$A$ must be a three-dimensional subspace of $\mathbb{R}^7$ (the vectors in Nul$A$ have 7 entries). Therefore, Nul$A \neq \mathbb{R}^3$.

Page 237, Problem 8:

Because there are four pivot columns, $\dim(\text{Col}A) = 4$, so $\dim(\text{Nul}A) = 8 - 4 = 4$. It is impossible for $\text{Col}A = \mathbb{R}^4$ because $\text{Col}A$ is a subspace of $\mathbb{R}^6$ (the vectors in $\text{Col}A$ have 6 entries).

Page 237, Problem 9:

Because $\dim(\text{Nul}A) = 3$ and $n = 6$, $\dim(\text{Col}A) = 6 - 3 = 3$. It is impossible for $\text{Col}A = \mathbb{R}^3$ because $\text{Col}A$ is a subspace of $\mathbb{R}^4$ (the vectors in $\text{Col}A$ have 4 entries).

Page 237, Problem 11:

Because $\dim(\text{Nul}A) = 3$ and $n = 5$, $\dim(\text{Row}A) = \dim(\text{Col}A) = 5 - 3 = 2$.

Page 237, Problem 18a:

True or False: If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

FALSE: As before, the pivot columns in B tell which columns of A form a basis for the column space of A.

Page 237, Problem 18c:

True or False: The dimension of the null space of A is the number of columns of A that are not pivot columns.

TRUE: Because the number of columns of A that are pivot columns equals the rank of A, by the Rank Theorem, the number of columns of A that are not pivot columns must be the dimension of the null space of A (see the proof of the Rank Theorem on page 233).

Page 238, Problem 31:

Compute $A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. Each column of this matrix is a multiple of $\mathbf{u}$, so

$\dim(\text{Col}A) = 1$, unless $a = b = c = 0$, in which case $\dim(\text{Col}A) = 0$. Because $\dim(\text{Col}A) = \text{rank}A$, $\text{rank}\mathbf{u}\mathbf{v}^{\text{T}} = \text{rank}A \leq 1$.

Page 238, Problem 32:

Notice that the second row of the matrix is twice the first. Therefore, take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, so that

$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$.