Homework 3
Answer key

Section 3.3 #1, 2, 4 | Section 3.4 #2, 4, 7, 9

Section 3.3

(1) a. $x_1 = 8 - 2x_4$
$x_2 = 6 - 3x_4$
$x_3 = 18 - 6x_4$

b. $0 < 8 - 2x_4 \Rightarrow x_4 \leq 4$
$0 < 6 - 3x_4 \Rightarrow x_4 \leq 2$
$0 < 18 - 6x_4 \Rightarrow x_4 \leq 3$

$\boxed{0 \leq x_4 \leq 2}$

c. x_2

d. x_2 should be extracted from the basis and replaced with x_4
pivot at the $3x_4$ term in the second equation

e. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1/3 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$

$x_1 - 2/3x_2 = 4$
$1/3x_2 + x_4 = 2$
$-2x_2 + x_3 = 6$

The associated BFS is $(4, 0, 6, 2)$ and is feasible

f. x_1: 4
x_2: 2
x_3: 3

The smallest ratio value occurs with the x_2 equation. Thus, we chose to replace x_2 in the set of basis variables

2) a. $\begin{bmatrix} 1 & 1 & 5 & 2 & 8 \\ 2 & 1 & 8 & 0 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 2 & 8 \\ 0 & -1 & -2 & -4 & -2 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 1 & 5 & 2 & 8 \\ 0 & 1 & 2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 & 6 \\ 0 & 1 & 2 & 4 & 2 \end{bmatrix}$

$$\boxed{\begin{aligned} x_1 \quad\quad + 3x_3 - 2x_4 &= 6 \\ x_2 + 2x_3 + 4x_4 &= 2 \end{aligned}}$$

minimize:

$(6 - 3x_3 + 2x_4) + (2 - 2x_3 - 4x_4) + 4x_3 + 7x_4 = z$

$$\boxed{-x_3 + 5x_4 = z - 8}$$

b. Since x_3 is negative in the objective function. Increasing its value will decrease the value of z.

x_1: $6/3 = 2$

x_2: $2/2 = 1$ ← minimum ∴ replace x_2

c. $\begin{bmatrix} 1 & 0 & 3 & -2 & 6 \\ 0 & 1 & 2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 & 6 \\ 0 & 1/2 & 1 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3/2 & 0 & -8 & 3 \\ 0 & 1/2 & 1 & 2 & 1 \end{bmatrix}$

$x_1 - 3/2 x_2 \quad - 8x_4 = 3$

$1/2 x_2 + x_3 + 2x_4 = 1$

minimize:

$-(1 - 1/2 x_2 - 2x_4) + 5x_4 = z - 8$

$-1 + 1/2 x_2 + 2x_4 + 5x_4 = z - 8$

$1/2 x_2 + 7x_4 = z - 7$

$$\boxed{\begin{aligned} &\text{BFS} \\ &(3, 0, 1, 0) \\ &z = 7 \end{aligned}}$$

(4) a) We want x_4 in the basis

$x_2 = 6 - 2x_4$ $\qquad$ $6 - 2x_4 > 0 \Rightarrow x_4 < 3$

$x_1 = 5 + x_4$ $\qquad$ $5 + x_4 > 0 \Rightarrow -5 < x_4$

Replace x_2 with x_4 in the set of B.V

$$\begin{bmatrix} 0 & 1 & -6 & 2 & 6 \\ 1 & 0 & 2 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1/2 & -3 & 1 & 3 \\ 1 & 0 & 2 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1/2 & -3 & 1 & 3 \\ 1 & 1/2 & -1 & 0 & 8 \end{bmatrix}$$

$1/2 x_2 - 3x_3 + x_4 = 3$

$x_1 + 1/2 x_2 - x_3 = 8$

minimize:

$4x_3 - 6(3 - 1/2 x_2 + 3x_3)$

$4x_3 - 18 + 3x_2 - 18x_3$

$3x_2 - 14x_3 = z + 18$

We have $z \boxed{+} 18$

b.) Our new canonical form and expression for z implies putting x_3 into the basis. However, the bounds on x_3 are

$x_4 = 3 + 3x_3 > 0$ $\qquad$ $\Rightarrow x_3 > 0$

$x_1 = 8 + x_3 > 0$ $\qquad$ $\Rightarrow x_3 > 0$

There is no limit on x_3 and therefore no lower bound on the objective function.

Section 3.4

(2) a) No pivots necessary $(5, 10, 0, 0)$ $z = 0$
b) No pivots necessary $(5, 10, 0, 0)$ $z = 0$
c) Unbounded below
d) Unbounded below
e) pivot at $2x_3$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & 2 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 5 \\ 0 & 1/2 & 1 & 0 & 5 \end{bmatrix}$$

$$x_1 \qquad -x_4 = 5$$
$$\tfrac{1}{2}x_2 + x_3 \qquad = 5$$

minimize: $-(5 - \tfrac{1}{2}x_2) + x_4 = z$

$$-5 + \tfrac{1}{2}x_2 + x_4 = z$$
$$\tfrac{1}{2}x_2 + x_4 = z + 5$$

$\boxed{(5, 0, 5, 0) \quad z = -5}$

f.) pivot at x_3 in first equation

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -2 & 1 & 0 & 2 & 10 \end{bmatrix}$$

$$x_1 + x_3 - x_4 = 0$$
$$-2x_1 + x_2 + 2x_4 = 10$$

$\boxed{(0, 10, 0, 0)}$

minimize: $-(-x_1 + x_4) + x_4 = z$

$$x_1 - x_4 + x_4 = z$$
$$x_1 = z$$

$\boxed{z = 0}$

g.) unbounded below

(4) minimize z where

$$x_1 + \ldots + a_{1,m+1}x_{m+1} + \ldots + a_{1n}x_n = b_1$$
$$+ x_2 + \ldots + a_{2,m+1}x_{m+1} + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$x_m + a_{m,m+1}x_{m+1} + \ldots + a_{mn}x_n = b_m$$
$$c_{m+1}x_{m+1} + \ldots + c_nx_n = z_0 + z$$
$$x_1, x_2, \ldots, x_n \geq 0$$

Let $c_s < 0$ and $b_r/a_{rs} = \min\{b_i/a_{is} : 1 \leq i \leq m \text{ and } a_{is} > 0\}$
Then we pivot at $a_{rs}x_s$ and in the resulting problem the

$$b_i^* = b_i - \frac{a_{is}b_r}{a_{rs}} \quad i = 1, \ldots, m; i \neq r \quad \text{and } b_r^* = b_r/a_{rs}$$

the objective function becomes

$$c_r^* x_r + c_{m+1}^* x_{m+1} \ldots + c_{s-1}^* x_{s-1} + c_{s+1}^* x_{s+1} + \ldots + c_n^* x_n$$
$$= z_0^* + z$$

the z_0^* is z_0 - the constant part of the $c_s x_s$ term which is $-c_s(b_r/a_{rs})$ so

$$z_0^* = z_0 - c_s b_r/a_{rs}$$

Then the new minimum of the objective function is

$\boxed{-z_0 + c_s b_r/a_{rs}}$

(7) If any $x_{m+1} \ldots x_n$ were not equal to 0 (and therefore >0 because $x_1 \ldots x_n$ must not be negative.
Then the $z = -z_0 + (c_{m+1}x_{m+1} \ldots c_nx_n)$ where
$c_{m+1}x_{m+1} \ldots c_nx_n > 0$
so the value of z would no longer be minimal
Then $x_{m+1} \ldots x_n$ must all be 0
Then clearly $x_1 \ldots x_m = b_1 \ldots b_m$

$\boxed{(b_1, \ldots, b_m, 0, \ldots 0) \text{ is a unique solution to the minimum value } -z_0}$

(9) If $z' = z$ for all solutions to the system of constraints then $x_{m+1} \dots x_n$ may not all be 0. Therefore, $z_0' = z_0$ and $c_j' = c_j$ for all j, $m+1 \leq j \leq n$ in order for this statement to be true for all solutions.

eg.

(1) $$c_{m+1} x_{m+1} + \dots + c_n x_n - z_0 = c'_{m+1} x_{m+1} + \dots + c_n' x_n - z_0'$$

$$x_{m+1}(c_{m+1} - c_{m+1}') + \dots + x_n(c_n - c_n') - (z_0 - z_0') = 0$$

$$\therefore \quad c_{m+1} - c_{m+1}' = 0 \qquad\qquad c_{m+1} = c_{m+1}'$$

$$\vdots \qquad\qquad \Rightarrow \qquad\qquad \vdots$$

$$c_n - c_n' = 0 \qquad\qquad c_n = c_n'$$

$$z_0 - z_0' = 0 \qquad\qquad z_0 = z_0'$$

If equation (1) is to be true for all solutions.