Homework #7
Answer Key

Section 4.5 #1, 2, 3(a-d)

1 max $x_1 + 2x_2$
subject to
$$2x_1 + x_2 \le 3$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

a) Determine the Dual Problem
minimize $3y_1 + 3y_2$
subject to
$$2y_1 + y_2 \ge 1$$
$$y_1 + 2y_2 \ge 2$$
$$y_1, y_2 \ge 0$$

b) Show that $x^* = (1,1)$ and $y^* = (0,1)$ are optimal solutions to the original and dual problems respectively, by using complementary slackness theorem

verify (1,1) is feasible for the max problem
$$2 + 1 = 3 \le 3$$
$$1 + 2 = 3 \le 3 \quad ✓$$
$$1, 1 \ge 0$$
Both constraints yield 0 slack so the optimal of the minimizing problem must be (y_1^*, y_2^*)
$$2y_1^* + y_2^* = 1$$
$$\underline{y_1^* + 2y_2^* = 2} \qquad y_1^* = 0$$
$$-3y_2^* = -3$$
$$y_2^* = 1$$

verify (0,1) is feasible for the dual
$$1 \ge 1$$
$$2 \ge 2 \quad ✓$$
$$0, 1 \ge 0$$
Both constraints have 0 slack $\therefore$ $x^* = (1,1)$ and $y^* = (0,1)$ are optimal solutions.

(2) max $2x_1 + 2x_2$

Subject to

$$\begin{aligned} x_1 \quad\quad + x_3 + x_4 &\le 1 \\ x_2 + x_3 - x_4 &\le 1 \\ x_1 + x_2 + 2x_3 \quad &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

a) Determine the Dual Problem

minimize $y_1 + y_2 + 3y_3$

Subject to

$$\begin{aligned} y_1 \quad\quad + y_3 &\ge 2 \\ y_2 + y_3 &\ge 2 \\ y_1 + y_2 + 2y_3 &\ge 0 \\ y_1 - y_2 \quad &\ge 0 \\ y_1, y_2, y_3, y_4 &\ge 0 \end{aligned}$$

b) $x^* = (1, 1, 0, 0)$

$\bar{x} \ge 0$

$1 + 0 + 0 = 1 \le 1$

$1 + 0 - 0 = 1 \le 1$

$1 + 1 + 0 + 0 = 2 \le 3$

$\boxed{\checkmark \text{ feasible}}$

$y^* = (1, 1, 1)$

$\bar{y} \ge 0$

$1 + 1 = 2 \ge 2$

$1 + 1 = 2 \ge 2$

$1 + 1 + 2 = 4 \ge 0$

$1 - 1 = 0 \ge 0$

$\boxed{\checkmark \text{ feasible}}$

c) Show that for this pair of solutions, for each j $x_j^* > 0$ implies the slack in the corresponding dual constraint is 0

$x^* = (1, 1, 0, 0)$ $\quad x_1^*, x_2^* > 0$

$x_3^*, x_4^* = 0$

and the first and second constraints of the dual have 0 slack with y^*.

d) Show that y^* is not an optimal solution to the dual

at x^* $Z = 2(1) + 2(1) + 0 + 0 = 4$
at y^* $V = 1 + 1 + 3(1) = 5$
$4 \neq 5$
so x^* and y^* are not the optimal solutions

e) NO. The slack in the third constraint of the original problem is not 0. So to satisfy the complementary slackness theorem, y_3^* must equal 0 and $y^* = (1, 1, 1)$ does not satisfy this condition.

③ a-d Prove or Disprove using complementary slackness

a) $(1, 1, 0, 0)$ is an optimal solution point to the maximization of problem 2.
$(1, 1, 0, 0)$ is feasible (shown in 2b)
$\neq 0$ slack in the third constraint
$y_3^* = 0$
$y_1^* + y_3^* = 2$ $\quad y_1^* = 2$
$y_2^* + y_3^* = 2$ $\quad y_2^* = 2$

is $(2, 2, 0)$ feasible?
$2 \geq 2$
$2 \geq 2$ $\quad$ feasible ✓
$4 + 0 \geq 0$
$0 \geq 0$

so $(1, 1, 0, 0)$ is the optimal solution point to the maximization problem.

b) (0,4,0,2) is an optimal Solution to (4.5.8) on page 157?

is (0,4,0,2) feasible?

$-4 \quad +12 = 8 \quad \le 8$ √ feasible

$12 \quad +4 = 16 \quad \le 16$

$4 \quad +6 = 10 \quad \le 12$

Slack in third constraint

$y_3^* = 0$

$-y_1^* + 3y_2^* + \cancel{y_3^*}^{\,0} = 3$

$6y_1^* + 2y_2^* + \cancel{3y_3^*} = 22$

$20y_2^* = 40$

$y_2^* = 2$

$y_1^* = 3$

is (3,2,0) feasible for the dual?

$2(3) + 5(2) = 16 \ge 9$

$-3 + 3(2) = 3 \ge 3$ √ feasible

$2(3) + 2 = 8 \ge 5$

$6(3) + 4 = 22 \ge 22$

$\boxed{\text{So, } (0,4,0,2) \text{ is the optimal solution}}$

c) Dual →

minimize $18y_1 + 4y_2 + 14y_3$

S.t

$8y_1 + 2y_2 + y_3 \ge 5$

$-2y_1 + 4y_2 + 3y_3 \ge 16$

$3y_1 - 7y_2 + y_3 \ge -4$

$3y_2 - y_3 \ge -1$

$-2y_1 + y_2 + 2y_3 \ge 7$

$y_1, y_2, y_3 \ge 0$

is (3,0,1,0,5) feasible in the original problem?

$24 + 3 - 10 = 17 \le 18$

$6 - 7 + 5 = 4 \le 4$ √ feasible

$3 + 1 + 10 = 14 \le 14$

slack in the first constraint

$y_1^* = 0$

$\cancel{8y_1^*} + 2y_2^* + y_3^* = 5$

$\cancel{3y_1^*} - 7y_2^* + y_3^* = -4$

$\cancel{-2y_1^*} + y_2^* + 2y_3^* = 7$

$2y_2^* + y_3^* = 5$ $\qquad 9y_2^* = 9$

$-7y_2^* + y_3^* = -4$ $\qquad y_2^* = 1$

$y_2^* + 2y_3^* = 7$ $\qquad y_3^* = 3$

$(0,1,3)$ feasible for dual?

$2 + 3 = 5 \geq 5$

$4 + 9 = 13 \not\geq 16$ not feasible

$\boxed{\therefore (3,0,1,0,5) \text{ is not optimal}}$

d) Dual

maximize $3y^2$

S.t $\quad y_1 + 2y_2 \leq 5$

$2y_1 + 3y_2 \leq 8$

$-y_1 + y_2 \leq 4$

$y_1 - y_2 \leq 2$

$y_1, y_2 \geq 0$

is $(1,0,1,0)$ feasible to the original problem?

$1 - 1 = 0 \geq 0$

$2 + 1 = 3 \geq 3$ ✓feasible

no slack in any constraints

$y_1^* + 2y_2^* = 5$

$-y_1^* + y_2^* = 4$

$3y_2^* = 9$

$y_2^* = 3$

$y_1^* = -1$ $\qquad$ not feasible

$\boxed{\therefore (1,0,1,0) \text{ is not optimal}}$