MATH381

test #1, 09/08/16

Solutions

Total 100

Show all work legibly. **Name**:________________________

1. (20) If solutions exist, then find at least one point (x_1, x_2) that solves

$$\begin{cases} 2x_1 + x_2 \le 2 \\ x_1 + 2x_2 \le 2 \\ 3x_1 + 3x_2 \ge 4 \end{cases}$$

$(x_1, x_2) =$

Solution. The intersection of two lines $2x_1 + x_2 = 2$ and $x_1 + 2x_2 = 2$ is $\left(\frac{2}{3}, \frac{2}{3}\right)$. We note that $3\frac{2}{3} + 3\frac{2}{3} = 4$.

2. (40) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$. Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$.

- (20) Select basic variables and free variables.

  Solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 5 & 7 & 9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  Hence

$$\begin{cases} x_1 \qquad = 2 + x_3 \\ \qquad x_2 = -1 - 2x_3 \end{cases}$$

  basic variables are:$\{x_1, x_2\}$

  free variables are:$\{x_3\}$

- (10) Find the general solution for the system $A\mathbf{x} = \mathbf{b}$.

  Solution.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + x_3 \\ -1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

- (10) Describe the geometric object represented the solutions.

  Solution. The geometric object is a straight line.

3. (20) Let $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y} \in \mathbf{R}^m$, and A is an $m \times n$ matrix. If $\mathbf{y}^T A\mathbf{x} = 2$ compute $\mathbf{x}^T A^T \mathbf{y}$.

   Solution.

$$2 = \mathbf{y}^T A\mathbf{x} = \left(\mathbf{y}^T A\mathbf{x}\right)^T = \mathbf{x}^T A^T \mathbf{y}.$$

   $\mathbf{x}^T A^T \mathbf{y} =$

4. (20) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a set of three linearly dependent vectors in $\mathbf{R}^n$. Consider two linear combinations of these vectors:

$$\mathbf{a} = t_1\mathbf{u}_1 + \mathbf{t}_2\mathbf{u}_2 + t_3\mathbf{u}_3, \text{ and } \mathbf{b} = s_1\mathbf{u}_1 + \mathbf{s}_2\mathbf{u}_2 + s_3\mathbf{u}_3.$$

   True or False? The vectors **a** and **b** are linearly dependent.

   Solution. Consider the following three vectors in $\mathbf{R}^2$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   While the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are linearly dependent, the vectors

$$\mathbf{a} = \mathbf{u}_1, \text{ and } \mathbf{b} = \mathbf{u}_2$$

   are linearly independent.

   Mark one and explain.

   □ True □ False

5. (20) A library collection contains m distinct books. One has to select n books from the library collection. Compute the number of all different selections.

   (Example. Suppose three books are available in the library. They are: "Pinocchio", "War and Peace", and "1984". The different selections of two out of three existing books are

   (a) {"Pinocchio", "War and Peace"},

   (b) {"Pinocchio", "1984"},

   (c) {"War and Peace", "1984"}

Hence in this example there are three different selections of two books.)

Solution.

the number is: m choose n is ${}_mC_n = \dfrac{m!}{n!(m-n)!}$.