MATH221

Midterm #2, 02/31/16

Total 100

Solutions

Show all work legibly. **Name**:________________________

1. (25) Let $T : \mathbf{R}^1 \to \mathbf{R}^1$ be a linear transformation so that $T(2) = 4$. Compute $T(7)$.

   Solution. $T(7) = T(3.5 \times 2) = 3.5 \times T(2) = 3.5 \times 4 = 14$.

   $T(7) = 14$

2. (25) Suppose a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$. True or False? T maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

   Solution. Let A be athe standard matrix of a linear transformation. Assume that T maps $\mathbf{R}^n$ onto $\mathbf{R}^n$. Thies yields existence of A^{-1}. If $\mathbf{y} = A\mathbf{u} = T(\mathbf{u}) = T(\mathbf{v}) = A\mathbf{v}$, then $A^{-1}\mathbf{y} = \mathbf{u} = \mathbf{v}$. This contradiction shows that the assumption is false, and completes the proof.

   Mark one and explain.

   □ True □ False

3. (25) Define a transformation $T : \mathbf{P}_2 \to \mathbf{R}^2$ by $T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$.

   (a) (10) True or False? T is a linear transformation.

   Solution.

$$T(c_1p_1 + c_2p_2) = \begin{bmatrix} c_1p_1(1) + c_2p_2(1) \\ c_1p_1(2) + c_2p_2(2) \end{bmatrix} = \begin{bmatrix} c_1p_1(1) \\ c_1p_1(2) \end{bmatrix} + \begin{bmatrix} c_2p_2(1) \\ c_2p_2(2) \end{bmatrix} = c_1T(p_1) + c_2T(p_2).$$

   Mark one and explain.

   □ True □ False

   (b) (15) Identify all polynomials $\mathcal{P}$ in $\mathbf{P}_2$ that vanish under T, i.e.

$$\mathcal{P} = \{p : p \in \mathbf{P}_2 \text{ and } T(p) = 0\}.$$

   Solution. If $p(x) = a_0 + a_1x + a_2x^2$, and $p(1) = p(2) = 0$, then

$$a_0 + a_1 + a_2 = 0 \text{ and } a_0 + 2a_1 + 4a_2 = 0.$$

That is $a_0 = 2a_2$, and $a_1 = -3a_2$. Hence $p(x) = t[2 - 3x + x^2]$.

$\mathcal{P} = \{t[2 - 3x + x^2] \ : \ -\infty < t < \infty\}$.

4. (25) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a linearly independent set of vectors in $\mathbf{R}^n$. True or False? The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbf{R}^n$.

Solution. Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent the matrix $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is invertible. If $\mathbf{b} \in \mathbf{R}^n$, and $\begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} = \mathbf{c} = A^{-1}\mathbf{b}$, then $c_1\mathbf{a}_+ \ldots + c_n\mathbf{a}_n = A\mathbf{c} = \mathbf{b}$.

Mark one and explain.

□ True □ False