MATH221

Midterm #3, 05/03/16

Total 100

Solutions

Show all work legibly. **Name**:______________________

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

1. (20) Find eigenvalues λ_1 and λ_2 of the matrix A.

   Solution. $\det(A - \lambda I) = (2-\lambda)(1-\lambda)$. Hence $\lambda_1 = 2$, and $\lambda_2 = 1$.

   $\lambda_1 =$ $\qquad\qquad$ $\lambda_2 =$

2. (20) Find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

   Solution.

   - If $(A - \lambda_1 I)\mathbf{x} = 0$, then $\mathbf{x} = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ solves the equation.
   - If $(A - \lambda_2 I)\mathbf{x} = 0$, then $\mathbf{x} = \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ solves the equation.

   $\mathbf{v}_1 = \begin{bmatrix} \quad \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \quad \end{bmatrix}$

3. (20) Find the inverse V^{-1} of the matrix $V = [\mathbf{v}_1, \mathbf{v}_2]$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of the matrix A.

   Solution.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

   $V^{-1} =$

4. (20) Use V and V^{-1} to compute A^{10}.

   Solution.

$$A^{10} = V \begin{bmatrix} 2^{10} & 0 \\ 0 & 1 \end{bmatrix} V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^{10} & 1 - 2^{10} \\ 0 & 1 \end{bmatrix}.$$

   $A^{10} =$

5. (20) Use the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to build an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbf{R}^2$.

Solution. $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{|\mathbf{v}_1|} = \mathbf{v}_1$. If $\mathbf{y} = \mathbf{v}_2 + t\mathbf{w}_1$ such that $\mathbf{y}^T\mathbf{w}_1 = 0$, then $t = -1$, and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
The second orthonormal basis vector $\mathbf{w}_2$ is normalized $\mathbf{y}$. Since $|\mathbf{y}| = 1$ one has

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\mathbf{w}_1 = \begin{bmatrix} \quad \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} \quad \end{bmatrix}$