MATH 221, Spring 2016 - Homework 2 Solutions

Due Tuesday, February 16

Section 1.4

Page 40, Problem 2:

The product is **not defined** because the order of the matrix is 3x1 and the order of the vector is 2x1. The number of columns of the first matrix (3) does not equal the number of entries of the vector (2).

Page 40, Problem 4:

The product **is defined** because the order of the matrix is 2x3 and the vector is 3x1 (so the number of columns (3) in the matrix is equal to the number of entries in the vector). The order of the product should be 2x1, the number of rows of the matrix and the number of entries of the vector.

a. Using the definition, as in Example 1 on page 35:

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

b. Using the row-vector rule (explained on page 38):

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 3(2) + -4(1) \\ 3(1) + 2(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Page 40, Problem 6:

This exercise is similar to part a of the problem 4, which is like Example 1. Use the elements of the vector as scalars for the columns of the matrix:

$$-3 \cdot \begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5 \cdot \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

Page 40, Problem 8:

This is similar to the previous exercise, but now write the column vectors as a 2x4 matrix, the scalars as a 4x1 column-vector, and keep the left-side of the equation as a two-column vector:

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Page 40 Problem 9:

Vector Equation: $x_1\begin{bmatrix}5\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix}+x_3\begin{bmatrix}-3\\4\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix}$ Matrix Equation: $\begin{bmatrix}5&1&-3\\0&2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix}$

Page 40, Problem 12:

Augmented Matrix: $\begin{bmatrix}1&2&-1&1\\-3&-4&2&2\\5&2&3&-3\end{bmatrix}$ Row-Reduction: $\begin{bmatrix}1&2&-1&1\\0&2&-1&5\\0&-8&8&-8\end{bmatrix}\rightarrow\begin{bmatrix}1&2&-1&1\\0&2&-1&5\\0&1&-1&1\end{bmatrix}\rightarrow$

$\begin{bmatrix}1&2&-1&1\\0&2&-1&5\\0&0&1&3\end{bmatrix}\rightarrow\begin{bmatrix}1&2&0&4\\0&2&0&8\\0&0&1&3\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0&-4\\0&1&0&4\\0&0&1&3\end{bmatrix}$ The solution, as a vector: $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-4\\4\\3\end{bmatrix}$

Page 40, Problem 13:

To answer this question, determine if **u** is in the Span of these columns, determine if **u** is a linear combination of the columns of A. That is, determine if A$\mathbf{x}=\mathbf{u}$ has a solution. The augmented matrix is $\begin{bmatrix}3&-5&0\\-2&6&4\\1&1&4\end{bmatrix}$

and row-reduction yields: $\begin{bmatrix}1&1&4\\3&-5&0\\-2&6&4\end{bmatrix}\rightarrow\begin{bmatrix}1&1&4\\0&-8&-12\\0&8&12\end{bmatrix}\rightarrow\begin{bmatrix}1&1&4\\0&2&3\\0&0&0\end{bmatrix}$.

Because there is no pivot in the last column, a solution exists, so **u is in the plane in $\mathbb{R}^3$ spanned by the columns of A**.

Page 41, Problem 35:

Assume A$\mathbf{y}=\mathbf{z}$ is true. Then, $5\mathbf{z}=5\text{A}\mathbf{y}=\text{A}(5\mathbf{y})$ (by Theorem 5b on page 39). Let $\mathbf{x}=5\mathbf{y}$. Then, A$\mathbf{x}=5\mathbf{z}$ is also consistent.

Section 1.5

Page 47, Problem 2:

Use row operations on the augmented matrix: $\begin{bmatrix}1&-2&3&0\\-2&-3&-4&0\\2&-4&9&0\end{bmatrix}\rightarrow\begin{bmatrix}1&-2&3&0\\0&-7&2&0\\0&-7&5&0\end{bmatrix}$

$\rightarrow\begin{bmatrix}1&-2&3&0\\0&-7&2&0\\0&0&3&0\end{bmatrix}$. Because there is a pivot in every column of the coefficient matrix, there are no free variables, **so the system has only the trivial solution**.

Page 47, Problem 8:

In order to solve this problem, put the matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0} \end{bmatrix}$ (where $\mathbf{a}_1$, etc. are the columns of A)

in reduced echelon form: $\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$, which is equivalent to the

system $\begin{matrix} x_1 - 2x_3 - 7x_4 = 0 \\ x_2 + 2x_3 - 4x_4 = 0 \end{matrix}$. It is clear that the basic variables are x_1 and x_2 while the free varaibles are x_3

and x_4. Solving for the free variables results in: $\begin{matrix} x_1 = 2x_3 + 7x_4 \\ x_2 = -2x_3 + 4x_4 \end{matrix}$. Writing in parametric vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Page 47, Problem 10:

This is the same process as problem 8 in this section: $\begin{bmatrix} -1 & -4 & 0 & -4 & 0 \\ 2 & -8 & 0 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{matrix} x_1 = -4x_4 \\ x_2 = 0 \end{matrix}$. The basic variables are x_1 and x_2 while the free variables are x_3 and x_4. The parametric vector

form is: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Page 47, Problem 12:

This is the same process as the previous two problems: $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1 = 2x_2 - 3x_3 - 29x_5 \\ x_4 = -4x_5 \\ x_6 = 0 \end{matrix}$. The basic variables are x_1, x_4, and x_6 .

The free variables are x_2, x_3, and x_5. The solution in parametric vector form is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Page 47, Problem 15:

First, realize that the second equation is the first equation shifted by 2. Solving the first equation for x_1 results in

$x_1 = -5x_2 + 3x_3$. In vector form, this is the same as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, which is a plane

through the origin spanned by $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. The solution to the second equation is:

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, which is a parallel plane through $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ instead of $\mathbf{0}$.

Page 47, Problem 18:

The system as an augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix}$ and row reduction yields: $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the parametric solution being $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$.

This solution is a line through $\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution to the homogenous equation in Exercise 6.

Page 48, Problem 35:

By inspection, the second column of A, $\mathbf{a}_2 = 3\mathbf{a}_1$. Therefore, one **nontrivial** (not $\mathbf{0}$) solution is

$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ or $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Page 48, Problem 38:

By Theorem 5b on page 39, $A(c\mathbf{w}) = cA\mathbf{w}$. Since $\mathbf{w}$ satisfies A$\mathbf{x} = \mathbf{0}$, A$\mathbf{w} = \mathbf{0}$. So, $cA\mathbf{w} = c\mathbf{0} = \mathbf{0}$, so $A(c\mathbf{w}) = \mathbf{0}$.

Section 2.1

Page 100, Problem 3:

To begin, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -3 & 5 \end{bmatrix}$ and

$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -15 \\ 9 & -6 \end{bmatrix}$

Page 100, Problem 5:

a. $A\mathbf{b}_1 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 26 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ -19 \end{bmatrix}$ So, $AB = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1(4)+3(-2) & -1(-2)+3(3) \\ 2(4)+4(-2) & 2(-2)+4(3) \\ 5(4)+-3(-2) & 5(-2)+-3(3) \end{bmatrix} = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

Page 100, Problem 6:

a. $A\mathbf{b}_1 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 12 \\ 3 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 22 \\ -22 \\ -2 \end{bmatrix}$ So, $AB = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

b. $AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4(1)+-3(3) & 4(4)+-3(-2) \\ -3(1)+5(3) & -3(4)+5(-2) \\ 0(1)+1(3) & 0(4)+1(-2) \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

Page 100, Problem 12:

Because A is 2x2 and B is 2x2, our new matrix of all zeros will also be 2x2. Essentially, we want to solve

$\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ with non-zero columns. Multiplying these matrices results in a linear system:

$\begin{array}{r} 3a-6c=0 \\ 3b-6d=0 \\ -2a+4c=0 \\ -2b+4d=0 \end{array}$, which can be broken into two separate systems: $\begin{array}{r} 3a-6c=0 \\ -2a+4c=0 \end{array}$ and $\begin{array}{r} 3b-6d=0 \\ -2b+4d=0 \end{array}$.

Using row reduction, $\begin{bmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $a = 2c$ and $b = 2d$. Answers will vary.

An example is $c = 1$, $d = 1$ so $a = b = 2$: $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Page 101, Problem 24:

Remember, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $D = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix}$. By definition of matrix multiplication, the columns of AD are equivalent to $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$, respectively. In order for $AD = I_3$, the systems generated by $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$ must each have at least one solution. Since the columns of A span $\mathbb{R}^3$, each of theses systems do have at least one solution (see Theorem 4 in Section 1.4). So, the matrix D is found by selecting one of the solutions from each of the systems ($A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$) and using it as the columns of D.

Page 101, Problem 26:

Let $\mathbf{b} \in \mathbb{R}^{\mathrm{m}}$ be arbitrary ($\mathbf{b}$ is an m x 1 matrix or vector). Assume $AD = I_m$ is true. Then, multiplying by $\mathbf{b}$ yields $AD\mathbf{b} = I_m\mathbf{b}$, which implies $AD\mathbf{b} = \mathbf{b}$ (because $I_m$is essentially 1). Because the order of the matrices is defined, $A(D\mathbf{b}) = \mathbf{b}$ (by Theorem 2 of this section on page 97). The product $D\mathbf{b}$ is a vector which can be written as $\mathbf{x} = D\mathbf{b}$. So, $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$. By Theorem 4 in Section 1.4, since $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$, A has a pivot position in every row. Because each pivot is in a different column, A must have at least as many columns as rows.

Page 101, Problem 33:

Let A be an arbitrary matrix of order i x j: $A = \begin{bmatrix} a_{11} & \dots & a_{1j} \\ \vdots & \dots & \vdots \\ a_{i1} & \cdots & a_{ij} \end{bmatrix}$ and B of order j x k : $B = \begin{bmatrix} b_{11} & \dots & b_{1k} \\ \vdots & \dots & \vdots \\ b_{j1} & \cdots & b_{jk} \end{bmatrix}$.

The product AB is defined because the number of columns of A (j) equals the number of rows of B (j).

The product is: $AB = \begin{bmatrix} a_{11}b_{11} + \ldots + a_{1j}b_{j1} & \cdots & a_{11}b_{1k} + \ldots + a_{1j}b_{jk} \\ \vdots & \dots & \vdots \\ a_{i1}b_{11} + ... + a_{ij}b_{j1} & \cdots & a_{i1}b_{1k} + \ldots + a_{ij}b_{jk} \end{bmatrix}$ which is a matrix of order i x k.

It follows that $(AB)^T = \begin{bmatrix} a_{11}b_{11} + \ldots + a_{1j}b_{j1} & \cdots & a_{i1}b_{11} + ... + a_{ij}b_{j1} \\ \vdots & \dots & \vdots \\ a_{11}b_{1k} + \ldots + a_{1j}b_{jk} & \cdots & a_{i1}b_{1k} + \ldots + a_{ij}b_{jk} \end{bmatrix}$, which is a matrix of order k x i.

$B^T = \begin{bmatrix} b_{11} & \dots & b_{j1} \\ \vdots & \dots & \vdots \\ b_{1k} & \cdots & b_{jk} \end{bmatrix}$, which is of order k x j, and $A^T = \begin{bmatrix} a_{11} & \dots & a_{i1} \\ \vdots & \dots & \vdots \\ a_{1j} & \cdots & a_{ij} \end{bmatrix}$, which is of order j x i.

The product B^TA^T is: $\begin{bmatrix} b_{11}a_{11} + \ldots + b_{j1}a_{1j} & \dots & b_{11}a_{i1} + \ldots + b_{j1}a_{ij} \\ \vdots & \dots & \vdots \\ b_{1k}a_{11} + \ldots + b_{jk}a_{1j} & \cdots & b_{1k}a_{i1} + \ldots + b_{jk}a_{ij} \end{bmatrix}$, which is a matrix of order k x i.

Looking at $(AB)^T$ and B^TA^T, it is clear that the matrices are equivalent.