

MATH 221, Spring 2016 - Homework 4 Solutions

Due Tuesday, March 8

Section 1.8

Page 69, Problem 26a:

Referring to the figure on page 47, because $\mathbf{q} - \mathbf{p}$ is parallel to line M, and \mathbf{p} lies on M, a parametric equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$. Expanding this expression yields $\mathbf{x} = \mathbf{p} + t\mathbf{q} - t\mathbf{p} \Rightarrow \mathbf{x} = \mathbf{p} - t\mathbf{p} + t\mathbf{q} \Rightarrow \mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$.

Page 69, Problem 26b:

Because $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$, $T(\mathbf{x}) = T((1 - t)\mathbf{p} + t\mathbf{q})$, then by definition of linear transformations,

$$T((1 - t)\mathbf{p} + t\mathbf{q}) = T((1 - t)\mathbf{p}) + T(t\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q})$$

If \mathbf{p} and \mathbf{q} are distinct, then this equation is representative of the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$ (like the equation found in part a). Otherwise, $T(\mathbf{p}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p}) - tT(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$, which is a single point.

(the same is true for $T(\mathbf{q})$)

Page 69, Problem 27:

$$T(\mathbf{x}) = T(s\mathbf{u} + t\mathbf{v}) = sT(\mathbf{u}) + tT(\mathbf{v}) \text{ such that } s, t \in \mathbb{R}$$

The set of images is $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$. If $\{T(\mathbf{u}), T(\mathbf{v})\}$ is linearly independent, then $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $\{T(\mathbf{u}), T(\mathbf{v})\}$ is linearly dependent (one is a multiple of the other and not both zero), then $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

Page 69, Problem 30:

Because $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ span \mathbb{R}^n , then any $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = \alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n$, for constants $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

$$\text{Then, } T(\mathbf{x}) = T(\alpha_1\mathbf{v}_1 + \dots + \alpha_n\mathbf{v}_n) = \alpha_1T(\mathbf{v}_1) + \dots + \alpha_nT(\mathbf{v}_n) = \alpha_1\mathbf{0} + \dots + \alpha_n\mathbf{0} = \mathbf{0}.$$

Page 69, Problem 32:

If T were linear then $T(c\mathbf{x}) = cT(\mathbf{x})$. Use any counterexample to show this is not true.

$$T((0, 1)) = (-2, -4), \text{ but } T(-1 \cdot (0, 1)) = T((0, -1)) = (-2, 4) \neq -1 \cdot T((0, 1)) = (2, 4)$$

Section 1.9

Page 78, Problem 11:

The transformation maps $\mathbf{e}_1 \rightarrow \mathbf{e}_1 \rightarrow -\mathbf{e}_1$ and $\mathbf{e}_2 \rightarrow -\mathbf{e}_2 \rightarrow -\mathbf{e}_2$, which in matrix form is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

This is the same as a rotation through π radians because $\begin{bmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Since a linear transformation is completely determined by what it does to the columns of the identity matrix (Theorem 10 of this section), the rotation transformation has the same effect as T on every vector in \mathbb{R}^2 .

Page 78, Problem 15:

The matrix entries are the coefficients of the variables on the right-hand side of the equation: $\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Page 78, Problem 22:

In this problem, we will use the fact that $T(\mathbf{x}) = A\mathbf{x}$. Because $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, we know \mathbf{x} is a 2×1 vector and the matrix A must be 3×2 . Therefore the set up of the transformation should be of the form:

$T(\mathbf{x}) = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix}$. The missing entries of the matrix A are the coefficients of the variables

on the right-hand side of the equation. Therefore: $T(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Because we are looking for \mathbf{x} such that

$T(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, we solve the system: $\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Page 79, Problem 26:

In order to answer this question, note that the transformation matrix is $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$.

Because there are more columns than rows, the columns must be linearly dependent. Therefore, T is **not one-to-one**.

If we row-reduce the matrix, $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$, we see that there is a pivot in every row.

Thus, the columns of A span \mathbb{R}^2 . Hence, T is **onto**.

Page 79, Problem 34:

Using the hint, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$ be arbitrary vectors and let $c, d \in \mathbb{R}$ be arbitrary scalars.

Because S is linear, $T(S(c\mathbf{u} + d\mathbf{v})) = T(cS(\mathbf{u}) + dS(\mathbf{v}))$. Because T is linear, $T(cS(\mathbf{u}) + dS(\mathbf{v})) = cT(S(\mathbf{u})) + dT(S(\mathbf{v}))$.

Therefore, $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation.