MATH 221, Spring 2016 - Homework 5 Solutions

Due Tuesday, March 22

Section 2.3

Page 115, Problem 2:

$$A = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$$

Notice that $\mathbf{a}_2 = -\frac{1}{2}\mathbf{a}_1$ where $\mathbf{a}_i$ is the column vector of the matrix A. Thus, the columns are linearly dependent. By Theorem 8 of this section, the **matrix is singular (nonivertible).** Also, notice that the determinant is equal to 0. So, by Theorem 4 of the previous section, the matrix is singular.

Page 115, Problem 4:

$$A = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix} \; A^T = \begin{bmatrix} -5 & 0 & 1 \\ 1 & 0 & 4 \\ 4 & 0 & 9 \end{bmatrix}$$

Notice that the columns of A^T are linearly dependent because the zero vector is a member of the set.

Thus, A^T is singular (noninvertible). Hence A is singular (nonivertible), by Theorem 8.

Also, because A contains a row of zeros, it cannot be reduced to the identity matrix.

Therefore, by Theorem 8, it is signular (noninvertible).

Page 115, Problem 8:

$$A = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the matrix is in echelon form, it is clear that there is a pivot in every row.

Hence, the matrix is invertible by Theorem 8.

Page 115, Problem 11a:

True or False: If the equation A$\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

TRUE: Because (d) of Theorem 8 is true, (b) must also be true.

Page 115, Problem 11d:

True or False: If the equation A$\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

TRUE: Because (d) of Theorem 8 is false, (c) must also be false. An $n \times n$ matrix can never have more than n pivot positions, so it must have fewer than n.

Page 115, Problem 11e:

True or False: If A^T is not invertible, then A is not invertible.

TRUE: Because (l) of Theorem 8 is false, (a) must also be false.

Page 115, Problem 12a:

True or False: If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.

TRUE: Because (k) of Theorem 8 is true, (j) is also true. Because $AD = I$, $D = A^{-1}$, so $DA = A^{-1}A = I$.

Page 115, Problem 12b:

True or False: If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^{\mathrm{n}}$ into $\mathbb{R}^{\mathrm{n}}$, then the row reduced echelon form of A is I.

FALSE: In order for this to follow from Theorem 8, $\mathbf{x} \mapsto A\mathbf{x}$ must map $\mathbb{R}^{\mathrm{n}}$ **onto** $\mathbb{R}^{\mathrm{n}}$, not into.

Page 115, Problem 12c:

True or False: If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^{\mathrm{n}}$.

TRUE: Because (e) of Theorem 8 is true, (h) must also be true.

Page 115, Problem 21:

Notice that on page 112, in the paragraph at the end of the page, it says (g) in Theorem 8 could be rewritten as

"The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^{\mathrm{n}}$."

In problem 21, this statement is false, thus (h) of Theorem 8 must also be false, so the columns of C **do not span** $\mathbb{R}^{\mathrm{n}}$.

Page 115, Problem 27:

Assume AB is invertible. Then, by Theorem 8(k) of this section, there exists an $n \times n$ matrix W such that $ABW = I$.

By properties of matrices (and because the order is defined), $ABW = A(BW) = I$.

Because A is square, let $BW = D$. Thus, by Theorem 8(k), A is invertible.

Page 115, Problem 33:

Let $T(\mathbf{x}) = A\mathbf{x}$. $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Because $\det A = 35 - 36 = -1$, A is invertible.

Thus, by Theorem 9 of this section, T is invertible.

$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. $A^{-1} = \frac{1}{-1}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. Thus, $T^{-1}(\mathbf{x}) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$.

Page 115, Problem 39:

Because T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the standard matrix A is invertible, by Theorem 8 of this section.

Hence, by Theorem 9 of this section, T is invertible and A^{-1} is the standard matrix of T^{-1}.

Thus, by Theorem 8 of this section, the columns of A^{-1} are linearly independent and span $\mathbb{R}^n$.

By Theorem 12 in Section 1.9, this shows that T^{-1} is a one-to-one mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Section 4.1

Page 196, Problem 16:

It is clear that W is not a vector space because it can never contain the zero vector (the first entry is always 1).

Page 196, Problem 21:

The set H is a subspace of M_{2x2} because:

1) If $a = b = d = 0$, the zero vector is contained in the space.

Let $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ be two arbitrary matrices in H.

2) Then, $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, so H is closed under addition.

3) Let β be an arbitrary scalar. Then, $\beta\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} \beta a_1 & \beta b_1 \\ 0 & \beta d_1 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

So H is closed under scalar multiplication.

Page 196, Problem 22:

The set M_{2x4} is the set of all matrices of the form $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ where the entries are arbitrary.

This set is a subspace (as stated in the problem).

Let the matrix F be $F = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix}$ where the entries are fixed.

Page 196, Problem 22 (cont):

The set $H = \{A \in M_{2x4} : FA = 0\}$ is a subset of M_{2x4}. To show H is a subspace:

1) Because $F0 = 0$, $0 \in H$.

2) Let A_1 and A_2 be arbitrary matrices in H. Then, $F(A_1) = 0$ and $F(A_2) = 0$.

Because$F(A_1 + A_2) = F(A_1) + F(A_2) = 0 + 0 = 0$. Thus, $A_1 + A_2 \in H$, so H is closed under addition.

3) Let $A \in H$ and $c \in \mathbb{R}$ be arbitrary. Thus, $FA = 0$. So, $F(cA) = cFA = c(FA) = 0$.

Thus, $cA \in H$, so H is closed under scalar multiplication.

Page 197, Problem 32:

To show $H \cap K$ is a subspace, check the three conditions:

1) Because H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Thus, $\mathbf{0} \in H \cap K$.

2) Let $\mathbf{u} \in H \cap K$ and $\mathbf{v} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$ and $\mathbf{v} \in H$ and $\mathbf{v} \in K$.

Because H and K are subspaces, $\mathbf{u} + \mathbf{v} \in H$ and $\mathbf{u} + \mathbf{v} \in K$. Thus, $\mathbf{u} + \mathbf{v} \in H \cap K$.

3) Let $c \in \mathbb{R}$ and $\mathbf{u} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$.

Because H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{u} \in K$. Thus, $c\mathbf{u} \in H \cap K$.

An example in $\mathbb{R}^2$ to show $H \cup K$ is not always a subspace would be $H = \{(x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, y) : y \in \mathbb{R}\}$

(the x-axis and y-axis, respectively). Let $\mathbf{u} =(1, 0) \in H \cup K$ and $\mathbf{v} = (0, 1) \in H \cup K$.

Then, $\mathbf{u} + \mathbf{v} = (1, 1)$, which is not in H or in K, so it is not in $H \cup K$.

Thus, $H \cup K$ is not closed under addition and is therefore not a subspace.

Page 197, Problem 33a:

To show $H + K$ is a subspace:

1) Because H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Thus, $\mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{0} \in H + K$.

2) Let $\mathbf{x}, \mathbf{y} \in H + K$ be arbitrary.

Then, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in H$ and $\mathbf{v} \in K$ and $\mathbf{y} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s} \in H$ and $\mathbf{t} \in K$.

Thus, $\mathbf{x} + \mathbf{y} = (\mathbf{u} + \mathbf{v}) + (\mathbf{s} + \mathbf{t}) = (\mathbf{u} + \mathbf{s}) + (\mathbf{v} + \mathbf{t})$. But, $\mathbf{u} + \mathbf{s} \in H$ and $\mathbf{v} + \mathbf{t} \in K$ (because H and K are subspaces).

Thus, $\mathbf{x} + \mathbf{y} \in H + K$, so $H + K$ is closed under addition.

3) Let $\mathbf{x} \in H + K$ and $c \in \mathbb{R}$ be arbitrary. Then, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in H$ and $\mathbf{v} \in K$. Thus, $c\mathbf{x} = c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Because H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$. Thus, $c\mathbf{x} \in H + K$, so $H + K$ is closed under scalar multiplication.

Page 197, Problem 33b:

Because every vector in H can be written as a sum of itself and $\mathbf{0}$ (the zero vector in K and H), H is a subset of $H+K$.

Because H contains the zero vector and H is closed under addition and scalar multiplication (because H is a subspace of V), H is a subspace of $H+K$ (this arguement also applies to K, so K is also a subspace of $H+K$.

Section 4.2

Page 206, Problem 6:

Solve the equation $A\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So, a spanning set for the null space is $\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 206, Problem 9:

The system of equations can be rearranged to $\begin{matrix} p-3q-4s-0r=0 \\ 2p-0q-s-5r=0 \end{matrix}$. So the vectors in W are solutions to this system.

Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 2 (and hence a vector space).

Page 206, Problem 14:

Notice that $W = \mathit{Col}\,\mathrm{A}$ for $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix}$. Therefore, W is a subspce of $\mathbb{R}^3$ (and a vector space) by Theorem 3

(look at Example 4 of this section).

Page 206, Problem 27:

Let $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$. Then, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution to $A\mathbf{x}=\mathbf{0}$. Thus, $\mathbf{x} \in \mathit{Nul}\,\mathrm{A}$. Since $\mathit{Nul}\,\mathrm{A}$ is a subspace

of $\mathbb{R}^3$, it is closed under scalar multiplication. Therefore, $10\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also in $\mathit{Nul}\,\mathrm{A}$ (a solution to the system).

Page 206, Problem 28:

Let $A = \begin{bmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$. Because there is a solution to $A\mathbf{x}=\mathbf{b}$, $\mathbf{b} \in \mathit{Col}\,\mathrm{A}$. Since $\mathit{Col}\,\mathrm{A}$ is a subspace

of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus, $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in $\mathit{Col}\,\mathrm{A}$. So, the second system must also have

a solution.