MATH 221, Spring 2016 - Homework 6 Solutions

Due Tuesday, March 29

Section 4.3

Page 213, Problem 3:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$,which reduces to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there are only two pivot positions, **the set of vectors are neither linearly independent nor span** $\mathbb{R}^3$, thus **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

Page 213, Problem 8:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix}$. Because there are four columns, **the set cannot be linearly indpendent in** $\mathbb{R}^3$. Thus, **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

To determine if the set of vectors span $\mathbb{R}^3$, row-reduce the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because there is a pivot position in each row, **the set of vectors do span** $\mathbb{R}^3$**.**

Page 213, Problem 13:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1 and 2 (look at matrix B, which is in row echelon form). Use these columns from matrix A to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$. To find a basis for Nul$A$, write the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables (x_3 and x_4): $\mathbf{x} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis for Nul$A$ is $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 214, Problem 14:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1, 3, and 5 (look at matrix$B$, which is in row echelon form). Use these columns from matrix A to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix} \right\}$. To find a basis for Nul$A$, we need the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables (x_2 and x_4). Because matrix B is only in row echelon form, reduce it to reduced row echelon form:

$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Thus a basis for Nul$A$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Page 214, Problem 21b:

True or False: If $H = \text{Span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

FALSE: The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent.

Page 214, Problem 21c:

True or False: The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^{\text{n}}$.

TRUE: Because the matrix is invertible, the columns span $\mathbb{R}^{\text{n}}$ and are linearly independent (by the Invertible Matrix Theorem). Hence, the columns form a basis for $\mathbb{R}^{\text{n}}$.

Page 214, Problem 21d:

True or False: A basis is a spanning set that is as large as possible.

FALSE: A basis is a spanning set that is as small possible (read "Two Views of a Basis" on p. 212).

Page 214, Problem 22a:

True or False: A linearly independent set in a subspace H is a basis for H.

FALSE: In order to be a basis, the set must also span H (by definition).

Page 214, Problem 22b:

True or False: If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

TRUE: By the Spanning Set Theorem, removing linearly dependent vectors in S will still result in a spanning set (this new set is a subset of S). Because the new set will eventually only contain linearly independent vectors, the set will be a basis for V.

Page 213, Problem 22e:

True or False: If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col$A$.

FALSE: The pivot columns in B tell which columns in matrix A form the basis for Col$A$ (see the warning after Theorem 6 on page 212).

Page 214, Problem 25:

While it might seem that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a spanning set for H, it is not. Notice that H is a subset of Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, there are vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in H, such as $\mathbf{v}_1$ and $\mathbf{v}_3$ (the second and third elements of these vectors are not equal). Therefore, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span H, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be a basis for H.

Page 215, Problem 33:

The polynomials are linearly independent because neither can be written as a scalar multiple of the other. As polynomials in $\mathbb{P}_3$, they can be written as vectors: $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, which as a matrix that is row-reduced is:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, indicating the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution (hence, the columns are linearly independent).

Section 4.5

Page 229, Problem 3:

Any vector in the subspace can be written as $a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$. Thus, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$ spans the subspace. To determine if this set is linearly independent, solve the matrix equation $\begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 2 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The matrix reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the only solution is the trivial solution, so the columns are linearly independent. Therefore, a basis for the subspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$. Because there are three vectors in the basis, the dimension of the subspace is 3.

Page 229, Problem 8:

The equation can be rewritten as $a = 3b - c$. Thus, any vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in the subspace can be written as

$b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Thus, the set $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ spans the subspace. It is clear that the set is linearly independent, but to verify that, reduce the matrix formed by the column vectors

$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which shows the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, so the columns are linearly independent. Thus, a basis is $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ with dimension 3.

Page 229, Problem 10:

Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 15 \end{bmatrix}$. It is clear that the set of these vectors is linearly dependent because $\mathbf{v}_2 = -2\mathbf{v}_1$ and $\mathbf{v}_3 = -3\mathbf{v}_1$. By the Spanning Set Theorem, the set $\{\mathbf{v}_1\}$ still spans $\mathbb{R}^2$ and because the set is linearly independent, it is also a basis for $\mathbb{R}^2$, so the dimension is 1.

Page 229, Problem 14:

Because there are three free variables, the dimension of Nul$A$ is 3 and because there are four pivot positions, the dimension of Col$A$ is 4.

Page 229, Problem 15:

Because there are two free variables, the dimension of Nul$A$ is 2 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 17:

Because there are no free variables, the dimension of Nul$A$ is 0 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 19a:

True or False: The number of pivot columns of a matrix equals the dimension of its column space.

TRUE: This is stated in the box on page 228 before Example 5.

Page 229, Problem 19d:

True or False: If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Page 229, Problem 20d:

True or False: If $\dim V = n$ and if S spans V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Section 4.6

Page 236, Problem 2:

Because $\text{rank}A = \dim(\text{Col}A)$, and since there are 3 pivot positions, $\text{rank}A = 3$. Because A is a 4×5 matrix, $\dim(\text{Nul}A) + \text{rank}A = 5$. Thus, $\dim(\text{Nul}A) = 5 - 3 = 2$. The basis for $\text{Col}A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$ and the basis for $\text{Row}A$ is the set of non-zero **rows** of B: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$. To find the basis for $\text{Nul}A$, reduce the matrix B to reduced-echelon form to find the solutions to the trivial equation:

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$ So the basis for $\text{Nul}A$ is: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Page 236, Problem 3:

For the same reasons problem 4, $\text{rank}A = 3$ and $\dim(\text{Nul}A) = 3$. The basis for $\text{Col}A$ is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ and the basis for $\text{Row}A$ is $\left\{ \begin{bmatrix} 2 \\ 6 \\ -6 \\ 6 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. Reducing B results in

Page 236, Problem 3 (cont):

$$\begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which implies } \mathbf{x} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the basis for Nul$A$ is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 237, Problem 7:

Because A is a 4×7 matrix, Col$A$ must be a subspace of $\mathbb{R}^4$. Since there are 4 pivot positions, it must be that Col$A = \mathbb{R}^4$.

Nul$A$ must be a three-dimensional subspace of $\mathbb{R}^7$ (the vectors in Nul$A$ have 7 entries). Therefore, Nul$A \neq \mathbb{R}^3$.

Page 237, Problem 8:

Because there are four pivot columns, dim(Col$A$) = 4, so dim(Nul$A$) $= 8 - 4 = 4$. It is impossible for Col$A = \mathbb{R}^4$ because Col$A$ is a subspace of $\mathbb{R}^6$ (the vectors in Col$A$ have 6 entries).

Page 237, Problem 9:

Because dim(Nul$A$) = 3 and $n = 6$, dim(Col$A$) $= 6 - 3 = 3$. It is impossible for Col$A = \mathbb{R}^3$ because Col$A$ is a subspace of $\mathbb{R}^4$ (the vectors in Col$A$ have 4 entries).

Page 237, Problem 11:

Because dim(Nul$A$) = 3 and $n = 5$, dim(Row$A$) = dim(Col$A$) $= 5 - 3 = 2$.

Page 237, Problem 18a:

True or False: If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

FALSE: As before, the pivot columns in B tell which columns of A form a basis for the column space of A.

Page 237, Problem 18c:

True or False: The dimension of the null space of A is the number of columns of A that are not pivot columns.

TRUE: Because the number of columns of A that are pivot columns equals the rank of A, by the Rank Theorem, the number of columns of A that are not pivot columns must be the dimension of the null space of A (see the proof of the Rank Theorem on page 233).

Page 238, Problem 31:

Compute $A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. Each column of this matrix is a multiple of $\mathbf{u}$, so

$\dim(\text{Col}A) = 1$, unless $a = b = c = 0$, in which case $\dim(\text{Col}A) = 0$. Because $\dim(\text{Col}A) = \text{rank}A$, $\text{rank}\mathbf{u}\mathbf{v}^{\text{T}} = \text{rank}A \leq 1$.

Page 238, Problem 32:

Notice that the second row of the matrix is twice the first. Therefore, take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, so that

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}.$$