MATH410

Midterm #1, 03/10/16

Total 100

Solutions

Show all work legibly. **Name**:______________________

1. (25) Let b and c be real numbers. Suppose $z_1 = 1+i$ is a root of the quadratic equation $z^2+bz+c=0$.

   - (20) Find the second root z_2.

     Solution.

     $$0 = z_1^2 + bz_1 + c = \overline{z_1^2 + bz_1 + c} = \overline{z_1}^2 + b\overline{z_1} + c.$$

     This shows that $\overline{z_1} = 1 - i$ is also a root of $z^2 + bz + c = 0$.

   - (5) Determine b and c.

     Solution. $z^2 + bz + c = (z - [1+i])(z - [1-i]) = z^2 - 2z + 2$.

     $b =$ $c =$

2. (25) Let D be a set of complex numbers z that satisfy

   $$|z-1| \le 1 \text{ and } |z+1| \le 1.$$

   True or False? The set D is star–shaped.

   Solution. The set D consists of a single element 0, hence it is star–shaped.

   Mark one and explain.

   □ True □ False

3. (25) Suppose that $\sum_{n=1}^{\infty} z_n$ converges. True or False? $\lim_{n\to\infty} z_n = 0$.

   Solution. Let $s_k = \sum_{n=1}^{k} z_n$, denote $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} z_n$ by s_0. For a given $\epsilon > 0$ we would like to identify N so that for each $n \ge N$ one has $|z_n| < \epsilon$. Since the sequence $\{s_k\}$ converges to s_0 there is N so that $|s_n - s_0| < \frac{\epsilon}{2}$ when $n \ge N$. In particular

   $$|z_{n+1}| = |s_n - s_{n+1}| = |s_n - s_0 + s_0 - s_{n+1}| \le |s_n - s_0| + |s_0 - s_{n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence for $n \geq N+1$ one has $|z_n| < \epsilon$.

Mark one and explain.

□ True □ False

4. (25) Suppose that the sequence $\{w_n\}$ converges. Let $|z| < 1$. True or False? The series $\sum_{n=1}^{\infty} w_n z^n$ converges.

Solution. Let $w = \lim_{n \to \infty} w_n$. There is N so that $|w - w_n| < 1$ if $n > N$. Let $M_1 = |w| + 1$, and $M_2 = \max\{|w_1|, \ldots, |w_N|\}$. If $M = \max\{M_1, M_2\}$, then $|w_n| < M$ for $n = 1, 2, \ldots$.
Note that $\sum_{k=1}^{n} |w_k z^k| < M \sum_{k=1}^{n} |z|^k \leq M \frac{1}{1-|z|}$. This implies convergence of the series $\sum_{k=1}^{\infty} |w_k z^k|$, and $\sum_{n=1}^{\infty} w_n z^n$.

Mark one and explain.

□ True □ False