MATH 225, FALL 2017 - HOMEWORK #1

Due Thursday, September 14

Section 2.2, page 43: 7, 10, 16, 25, 27(a)
Section 2.3, page 51: 8, 14, 18, 22, 28(c), 29

Section 2.2

Problem 7: $\frac{dx}{dt} = 3xt^2 \Rightarrow \frac{dx}{3x} = t^2 dt$. Integrating both sides results in $\frac{\ln x}{3} = \frac{t^3}{3} + C \Rightarrow \ln x = t^3 + \frac{C}{3} \Rightarrow x = e^{t^3 + C/3} = e^{t^3} \cdot e^{C/3} = Ce^{t^3}$ (because $e^{C/3}$ is a constant term).

Problem 10: $\frac{dx}{dt} = \frac{t}{xe^{t+2x}} \Rightarrow \frac{dx}{dt} = \frac{t}{xe^t \cdot e^{2x}} \Rightarrow xe^{2x}dx = \frac{t}{e^t}dt$ Next, compute the integral of both sides. To compute $\int xe^{2x}dx$, use integration by parts: $\int u \cdot dv = uv - \int v \cdot du$. In this case, $u = x$, $du = dx$, and $dv = e^{2x}$ so that $v = \frac{e^{2x}}{2}$. Then, $\int xe^{2x}dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$. To compue $\int \frac{t}{e^t}dt$ use integration by parts where $u = t$, $du = dt$, and $dv = \frac{1}{e^t}$ so that $v = -\frac{1}{e^t}$. Then, $\int \frac{t}{e^t}dt = -\frac{t}{e^t} - \int -\frac{1}{e^t}dt = -\frac{t}{e^t} - \frac{1}{e^t} + C$. Therefore, the equation becomes $\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} = -\frac{t}{e^t} - \frac{1}{e^t} + C$. At this point, is impossible to solve for $x$in terms of t, so the implicit form is sufficient.

Problem 16: $(x + xy^2)dx + e^{x^2}ydy = 0 \Rightarrow e^{x^2}ydy = -(x + xy^2)dx \Rightarrow \frac{dy}{dx} = -\frac{x(1+y^2)}{e^{x^2}y} \Rightarrow \frac{y}{1+y^2}dy = -\frac{x}{e^{x^2}}dx$ Next, compute the integral of both sides. Use u-substitution for $\int \frac{y}{1+y^2}dy$ by letting $u = 1 + y^2$ so $du = 2ydy$ and the integral becomes $\int \frac{du}{2u} = \frac{1}{2}\ln u$. Then $\int \frac{y}{1+y^2}dy = \frac{1}{2}\ln(1 + y^2)$. To find the integral of $-\frac{x}{e^{x^2}}dx$, use u-substitution and let $u = x^2$ so that $du = 2xdx$ and $\int -\frac{du}{2e^u} = \frac{1}{2e^u}$. Therefore, $\int -\frac{x}{e^{x^2}}dx = \frac{1}{2e^{x^2}} + C$. Equating these two expressions results in $\frac{1}{2}\ln(1 + y^2) = \frac{1}{2e^{x^2}} + C$. Solving for y results in $\ln(1 + y^2) = e^{-x^2} + 2C \Rightarrow 1 + y^2 = e^{e^{-x^2}+2C} \Rightarrow y = \sqrt{Ce^{e^{-x^2}} - 1}$.

Problem 25: First, solve the differential equation. $\frac{dy}{dx} = x^2(1+y) \Rightarrow \frac{dy}{1+y} = x^2dx \Rightarrow \ln(1+y) = \frac{x^3}{3} + C \Rightarrow y = Ce^{x^3/3} - 1$. Then, plug in the initial value: $3 = y(0) = C - 1 \Rightarrow C = 4$. So, the solution is $y(x) = 4e^{x^3/3} - 1$.

Problem 27a: First, separate the equation: $\frac{dy}{dx} = e^{x^2} \Rightarrow dy = e^{x^2}dx$. Then, integrate from $x = 0$ to $x = x_1$. So, this becomes$\int_0^{x_1} e^{x^2}dx = \int_0^{x_1} dy = y \mid_0^{x_1} = y(x_1) - y(0)$. Knowing the initial value $y(0) = 0$, this equation can be solved by letting $x_1 = x$ and changing the variable of integration to t to obtain $y(x) = \int_0^x e^{t^2}dt$.

Section 2.3

Problem 8: $\frac{dy}{dx} - y - e^{3x} = 0$ in standard form is $y' - y = e^{3x}$ so that the integrating factor is $\mu(x) = e^{\int -1dx} = e^{-x}$. Multiplying the integrating factor on both sides of the equation results in $(y' - y)e^{-x} = e^{3x}e^{-x} \Rightarrow (e^{-x}y)' = e^{2x} \Rightarrow e^{-x}y = \frac{1}{2}e^{2x} + C \Rightarrow y(x) = \frac{1}{2}e^{3x} + Ce^x$.

Problem 14: $x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}$ in standard form is $y' + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x$ so that the integrating factor is $\mu(x) = e^{\int \frac{3}{x}dx} = e^{\ln x^3} = x^3$. (NOTE: by the definition of exponentials and logarithms as inverse functions, $e^{\ln f(x)} = f(x)$). Multiplying the integrating factor on both sides of the equation results in $(y' + \frac{3}{x}y)x^3 = x\sin x - 3x^4 \Rightarrow (x^3y)' = x\sin x - 3x^4$. Integrating both sides results in $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + C$ (NOTE: $\int x\sin x$ can be computed using integration by parts). Solving for y yields the solution $y(x) = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} - \frac{3}{5}x^2 + \frac{C}{x^3}$.

Problem 18: First, solve the differential equation by putting $\frac{dy}{dx} + 4y - e^{-x} = 0$ in standard form: $y' + 4y = e^{-x}$ so that the integrating factor is $\mu(x) = e^{\int 4dx} = e^{4x}$. Next, multiply both sides of the equation by the integrating factor to arrive at $(e^{4x}y)' = e^{3x}$. Then, integrate both sides of the equation to get $e^{4x}y = \frac{1}{3}e^{3x} + C$. Therefore, the general solution is $y(x) = \frac{1}{3}e^{-x} + Ce^{-4x}$. Using the initial value $y(0) = \frac{4}{3}$, solve for C: $\frac{4}{3} = y(0) = \frac{1}{3} + C \Rightarrow C = 1$. Therefore, the solution is $y(x) = \frac{1}{3}e^{-x} + e^{-4x}$.

Problem 22: First, solve the differential equation by putting $\sin x \frac{dy}{dx} + y\cos x = x\sin x$ in standard form: $y' + \frac{\cos x}{\sin x}y = x$. Notice that $\frac{\cos x}{\sin x} = \cot x$. The integrating factor is $\mu(x) = e^{\int \cot x dx} = e^{\ln(\sin x)} = \sin x$. (NOTE: $\int \cot x dx = \int \frac{\cos x}{\sin x}dx$ and letting $u = \sin x$, $du = \cos x dx$ so this becomes $\int \frac{du}{u} = \ln u$, which is $\ln(\sin x)$.) Multiplying both sides of the equation by $\sin x$ results in $(y \sin x)' = x\sin x$. Integrating both sides of the equation results in $y\sin x = \sin x - x\cos x + C$. Solving for y in terms of $x$gives the general solution $y(x) = 1 - x\cot x + \frac{C}{\sin x}$. Then, use the initial value $2 = y(\frac{\pi}{2}) = 1 + C \Rightarrow C = 1$. Therefore, the solution is $y(x) = 1 - x\cot x + \csc x$.

Problem 28c: Assume $\hat{y}(x)$ is a solution. Let C be a constant. Then, $\frac{d}{dx}(C\hat{y}(x)) = C\hat{y}'(x)$. The expression $C\hat{y}'(x) + P(x)(C\hat{y}(x))$ is equivalent to $C(\hat{y}'(x) + P(x)\hat{y}(x))$. Since $\hat{y}(x)$ is a solution, it follows that $\hat{y}'(x) + P(x)\hat{y}(x) = 0$, so that $C(\hat{y}'(x) + P(x)\hat{y}(x)) = C(0) = 0$. Therefore, $C\hat{y}(x)$ is also a solution.

Problem 29: Use the hint and reverse the independent and dependent variables so that it becomes $\frac{dx}{dy} = e^{4y} + 2x$. Then, putting this in standard form we get $x' - 2x = e^{4y}$, so the integrating factor is $\mu(x) = e^{\int -2dy} = e^{-2y}$. Multiplying both sides by this factor we get $(xe^{-2y})' = e^{2y}$ and integrating we get $xe^{-2y} = \frac{1}{2}e^{2y} + C$, so that $x(y) = \frac{1}{2}e^{4y} + Ce^{2y}$.