MATH 225, FALL 2017 - HOMEWORK #3

Due Thursday, September 28

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Section 2.6

Problem 1: The equation $2txdx + (t^2 - x^2)dt = 0$ can be rewritten as $\frac{dx}{dt} = \frac{x^2-t^2}{2tx} = \frac{1}{2}\left(\frac{x}{t} - \frac{t}{x}\right)$, which is a function of $\frac{x}{t}$, so the equation is homogeneous. Notice that the function can also be written as $\frac{dx}{dt} - \frac{1}{2t}x = -\frac{t}{2}x^{-1}$, so the equation is also a Bernoulli equation.

Problem 11: The equation $(y^2 - xy)dx + x^2dy = 0$ can be rewritten as $\frac{dy}{dx} = \frac{xy-y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$. Let $v = \frac{y}{x}$. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx} = v - v^2 \Rightarrow -\frac{dv}{v^2} = \frac{dx}{x}$. So, $\frac{1}{v} = \ln|x| + C$. Substituting $v = \frac{y}{x}$, we get $\frac{x}{y} = \ln|x| + C$ or $y = \frac{x}{\ln|x|+C}$. An additional solution is $x = 0$, $y = 0$.

Problem 13: The equation $\frac{dx}{dt} = \frac{x^2+t\sqrt{t^2+x^2}}{tx}$ can be rewritten as $\frac{dx}{dt} = \frac{x}{t} + \frac{\sqrt{t^2+x^2}}{x} = \frac{x}{t} + \frac{\sqrt{1+(\frac{x}{t})^2}}{\frac{x}{t}}$. Let $v = \frac{x}{t}$. Then , $\frac{dx}{dt} = v + t\frac{dv}{dt} = v + \frac{\sqrt{1+v^2}}{v}$. Thereore, $\frac{v}{\sqrt{1+v^2}}dv = \frac{dt}{t}$. The integral $\int \frac{v}{\sqrt{1+v^2}}dv$ can be computed using substitution, letting $u = 1 + v^2$ so $du = 2vdv$, then the integral is $\frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{1 + v^2}$. Therefore, the solution is $\sqrt{1 + v^2} = \ln|t| + C$. Substituting $v = \frac{x}{t}$, we get $\sqrt{1 + \left(\frac{x}{t}\right)^2} = \ln|t| + C$.

Problem 18: The function $\frac{dy}{dx} = (x+y+2)^2$ can be expressed with the substitution $z = x+y+2$, so that $\frac{dz}{dx} = 1+\frac{dy}{dx}$ and $\frac{dy}{dx} = \frac{dz}{dx} - 1$. Then, $\frac{dz}{dx} - 1 = z^2 \Rightarrow \frac{dz}{z^2+1} = dx \Rightarrow \tan^{-1}(z) = x+C$. Replacing z with $x+y+2$ yields $\tan^{-1}(x+y+2) = x+C$ so that $x + y + 2 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 2$.

Problem 22: For the equation $\frac{dy}{dx} - y = e^{2x}y^3$, divide by y^3 so the equation becomes $y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}$. Let $v = y^{-2}$ so that $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y^3}{-2}\frac{dv}{dx}$. Using this substitution, the equation becomes $-\frac{1}{2}\frac{dv}{dx} - v = e^{2x} \Rightarrow \frac{dv}{dx} + 2v = -2e^{2x}$. Then, the integrating factor is e^{2x} and the solution is $e^{2x}v = -\frac{1}{2}e^{4x} + C \Rightarrow v = -\frac{1}{2}e^{2x} + Ce^{-2x}$ or $y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}$ so that $y = \pm\sqrt{-\frac{2}{e^{2x}+2Ce^{-2x}}}$.

Problem 24: For the equation $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$, divide by $y^{1/2}$ so the equation becomes $y^{-1/2}\frac{dy}{dx} + \frac{\sqrt{y}}{x-2} = 5(x-2)$. Let $v = y^{1/2}$ so that $\frac{dv}{dx} = \frac{1}{2}\frac{1}{\sqrt{y}}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2\sqrt{y}\frac{dv}{dx}$. Using this substitution, the equation becomes $2\frac{dv}{dx} + \frac{v}{x-2} = 5(x-2) \Rightarrow \frac{dv}{dx} + \frac{1}{2(x-2)}v = \frac{5}{2}(x-2)$. Then, the integrating factor is $e^{\ln(x-2)^{1/2}} = \sqrt{x-2}$. Therefore, the solution is $\sqrt{x-2}v = \int \frac{5}{2}(x-2)^{3/2}dx = (x-2)^{5/2} + C \Rightarrow v = (x-2)^2 + \frac{C}{\sqrt{x-2}}$ or $y = \left((x-2)^2 + \frac{C}{\sqrt{x-2}}\right)^2$.

Problem 33: The equation $2txdx + (t^2 - x^2)dt = 0$ can be rewritten as $\frac{dx}{dt} - \frac{1}{2t}x = -\frac{t}{2}x^{-1}$. Then, divide by x^{-1} so the equation becomes $x\frac{dx}{dt} - \frac{1}{2t}x^2 = -\frac{t}{2}$. Then, let $v = x^2$ so that $\frac{dv}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{2x}\frac{dv}{dt}$. Using this substitution, the equation becomes $\frac{1}{2}v' - \frac{1}{2t}v = -\frac{t}{2} \Rightarrow v' - \frac{1}{t}v = -t$. Thus, the integrating factor is $e^{-\ln t} = \frac{1}{t}$ and the solution is $\frac{1}{t}v = -t + C$. Therefore, $x = \pm\sqrt{-t^2 + tC}$.

Problem 41: The equation $\frac{dy}{dx} = y-x-1+(x-y+2)^{-1}$ is from Example 2 on page 70. Using the substitution $v = x-y+2$, we get $\frac{dv}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$. Therefore, $1 - \frac{dv}{dx} = -v + 1 + v^{-1} \Rightarrow -\frac{dv}{dx} = -v + v^{-1} \Rightarrow \frac{dv}{v-v^{-1}} = dx$. Therefore, $\frac{1}{2}\ln(v^2 - 1) = x + C \Rightarrow \ln(v^2 - 1) = 2x + C \Rightarrow v^2 = Ce^{2x} + 1$ or $(x - y + 2)^2 = Ce^{2x} + 1$.

Problem 42: The substitution $y = vx^2$ is equivalent to $v = \frac{y}{x^2}$, so $\frac{dv}{dx} = -\frac{2y}{x^3} + \frac{dy}{dx}\frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2\frac{dv}{dx} + \frac{2y}{x}$. Then, the equation $\frac{dy}{dx} = \frac{2y}{x} + \cos(\frac{y}{x^2})$ becomes $x^2\frac{dv}{dx} + \frac{2y}{x} = \frac{2y}{x} + \cos(v) \Rightarrow x^2\frac{dv}{dx} = \cos v$. Therefore, $\sec v dv = \frac{dx}{x^2}$ so that $\ln(\tan v + \sec v) = -\frac{1}{x} + C$. Then, $\tan v + \sec v = Ce^{-1/x}$ or $\tan\left(\frac{y}{x^2}\right) + \sec\left(\frac{y}{x^2}\right) = Ce^{-1/x}$.