MATH 225, FALL 2017 - HOMEWORK #4

Due Thursday, October 12

Section 4.2, page 165: 7, 10, 15, 28, 34, 43
Section 4.3, page 173: 3, 8, 24, 31(a)

Section 4.2

Problem 7: The characteristic equation is $6r^2 + r - 2 = 0$ which has solutions $r = \frac{-1 \pm \sqrt{1-4(6)(-2)}}{2(6)} = -\frac{1}{12} \pm \frac{7}{12} \Rightarrow r = \frac{1}{2}$, $r = -\frac{2}{3}$. Therefore, the general solution to the differential equation is $y(x) = c_1 e^{x/2} + c_2 e^{-\frac{2}{3}x}$.

Problem 10: The characteristic equation is $r^2 - r - 11 = 0$ which has solutions $r = \frac{1 \pm \sqrt{1-4(1)(-11)}}{2(1)} = -\frac{1}{2} \pm \frac{3\sqrt{5}}{2}$. Therefore, the general solution to the differential equation is $y(x) = c_1 e^{(-\frac{1}{2}+\frac{3\sqrt{5}}{2})x} + c_2 e^{(-\frac{1}{2}-\frac{3\sqrt{5}}{2})x}$.

Problem 15: The characteristic equation is $r^2 - 4r - 5 = 0$ which has solutions $r = \frac{4 \pm \sqrt{16-4(1)(-5)}}{2(1)} = 2 \pm 3 \Rightarrow r = -1$, $r = 5$. Therefore, the general solution to the differential equation is $y(x) = c_1 e^{-x} + c_2 e^{5x}$. In order to apply the initial conditions, take the first derivative of y: $y'(x) = -c_1 e^{-x} + 5c_2 e^{5x}$. Then, apply the initial conditions to arrive at a linear system: $\begin{array}{c} y(-1) = c_1 e^1 + c_2 e^{5(-1)} = 3 \\ y'(-1) = -c_1 e^1 + 5c_2 e^{5(-1)} = 9 \end{array} \Rightarrow c_2 e^{-5} + 5c_2 e^{-5} = 12 \Rightarrow 6e^{-5}c_2 = 12 \Rightarrow c_2 = 2e^5$. Using this information, find $c_1 : c_1 e + (2e^5)e^{-5} = 3 \Rightarrow c_1 e + 1 = 3 \Rightarrow c_1 = e^{-1}$. Therefore, the solution to the initial value problem is $y(x) = 2e^{-1}e^{-x} + 2e^5 e^{5x} \Rightarrow y(x) = e^{-1-x} + 2e^{5+5x}$.

Problem 28: We want to determine if $e^{3t} = \alpha e^{-4t}$, where α is a constant, for all t in the interval (0, 1). If this were true, then $\alpha = e^{7t}$. But, e^{7t} is a different value for each t in (0, 1), so α is not a constant. Therefore, the functions are linearly independent on (0, 1).

Problem 34: For part a, recall that for a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is given by the equation $ad - bc$. Therefore, the determinant of $\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$ is $y_1(t)y_2'(t) - y_2(t)y_1'(t)$. For the first portion of part b, assume y_1 and y_2 are linearly independent on I. Then, by Lemma 1, the Wronskian must never be 0; otherwise the functions are linearly dependent. To prove the other direction, assume two differentiable functions $y_1(t)$ and $y_2(t)$ are linearly dependent on I. Then, for some constant α, $y_2(t) = \alpha y_1(t)$, so that $W[y_1(t), y_2(t)] = y_1(t)y_2'(t) - y_2(t)y_1'(t) = y_1(t)(\alpha y_1(t)) - \alpha y_1(t)y_1'(t) = 0$. Therefore, it must also be true that if the Wronskian is never zero on I then y_1 and y_2 are linearly independent.

Problem 43: The characteristic equation is $r^3 - r = 0$ which can be written as $r(r+1)(r-1) = 0$, so $r = 0$, $r = -1$, $r = 1$ are solutions. Therefore, the general solution is $y(x) = a + be^{-x} + ce^x$. To solve the IVP, take the first and second derivatives: $y'(x) = -be^{-x} + ce^x$ and $y''(x) = be^{-x} + ce^x$. Then, apply the initial values: $\begin{array}{c} y(0) = a + b + c = 2 \\ y'(0) = -b + c = 3 \\ y''(0) = b + c = -1 \end{array}$. Solving the system of equations yields $c = 1$, $b = -2$, $a = 3$. Therefore, the general solution is $y(x) = 3 - 2e^{-x} + e^x$.

Section 4.3

Problem 3: The characteristic equation is $r^2 - 10r + 26 = 0$ which has solutions $r = \frac{10 \pm \sqrt{100-4(1)(26)}}{2(1)} = 5 \pm \frac{\sqrt{-4}}{2} = 5 \pm i$. Therefore, the general solution is $y(x) = c_1 e^{5x}\cos(x) + c_2 e^{5x}\sin(x)$.

Problem 8: The characteristic equation is $4r^2 - 4r + 26 = 0$ which has solutions $r = \frac{4 \pm \sqrt{16-4(4)(26)}}{2(4)} = \frac{1}{2} \pm \frac{\sqrt{-400}}{8} = \frac{1}{2} \pm \frac{5}{2}i$. Therefore, the general solution is $y(x) = c_1 e^{x/2}\cos(\frac{5}{2}x) + c_2 e^{x/2}\sin(\frac{5}{2}x)$.

Problem 24: The characteristic equation is $r^2 + 9 = 0$ which has solutions $r = \pm\sqrt{-9} = \pm 3i$. Therefore, the general solution is $y(x) = c_1 \cos(3x) + c_2 \sin(3x)$. To solve the IVP, take the first derivative: $y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x)$. Then, apply the initial values: $\begin{array}{c} y(0) = c_1 = 1 \\ y'(0) = 3c_2 = 1 \end{array} \Rightarrow \begin{array}{c} c_1 = 1 \\ c_2 = \frac{1}{3} \end{array}$. So, the solution to the IVP is $y(x) = \cos(3x) + \frac{1}{3}\sin(3x)$.

Problem 31: In this problem, since $b = 0$, we should expect oscillations and no damping effect (see the paragraph on page 171). The characteristic equation is $r^2 + 16 = 0$ which has solutions $r = \pm\sqrt{-16} = \pm 4i$. Therefore, the general solution is $y(t) = c_1 \cos(4t) + c_2 \sin(4t)$. To solve the IVP, take the first derivative: $y'(t) = -4c_1 \sin(4x) + 4c_2 \cos(4x)$. Then, apply the initial values: $\begin{array}{c} y(0) = c_1 = 2 \\ y'(0) = 4c_2 = 0 \end{array} \Rightarrow \begin{array}{c} c_1 = 2 \\ c_2 = 0 \end{array}$. So, the solution to the IVP is $y(t) = 2\cos(4t)$. As t approaches infinity, the system oscillates, as predicted.