MATH 225, FALL 2017 - HOMEWORK #6

Due Thursday, November 2

Section 4.7, page 200: 32, 34, 47, 48
Section 7.2, page 360: 2, 3, 12, 13, 17, 30

Section 4.7

Problem 32: The Wronskian is given by $W = y_1y_2' - y_1'y_2$. Therefore, $W' = y_1'y_2' + y_1y_2'' - (y_1'y_2' + y_1''y_2) = y_1y_2'' + y_1''y_2$. Therefore, $W' + pW = y_1y_2'' + y_1''y_2 + p(y_1y_2' - y_1'y_2) = y_1(y_2'' + py_2') - y_2(y_1'' + py_1') = y_1(-qy_2) - y_2(-qy_1) = 0$. The equation is given as $\frac{dW}{W} = -p\,dt$. Solving this equation amounts to proving the formula.

Problem 34: To find a solution to the homogeneous equation, you need to use the Superposition Principle (see Exercise 30). We know that we are look for a solution to $y'' + p(t)y' + q(t)y = g(t) - g(t) = 0$. Therefore, one solution (by the Superposition Principle) is $y_1(t) = t^2 - t$ and a second is $y_2(t) = t^3 - t$. You can verify that these solutions are linearly independent. The solution to the original problem is the sum of the general solution to the homogeneous equation and a particular solution. Therefore, $y(t) = t + c_1(t^2 - t) + c_2(t^3 - t)$. To solve the initial value problem, differentiate y and then use the initial values to arrive at a system of equations. The final answer is $y(t) = 2t^3 - 6t^2 + 5t$. To find $p(t)$, we use Abel's formula to arrive at $\frac{W'}{W} = -p$. Then, we compute $W(y_1, y_2) = (t^2 - t)(3t^3 - 1) - (t^3 - t)(2t - 1) = t^2(t-1)^2$ and $W' = 2t(t-1)(2t-1)$. So, $p(t) = -\frac{2t(t-1)(2t-1)}{t^2(t-1)^2}$.

Problem 47: Use the formula for reduction of order on page 198. In order to use this formula, the equation must be in standard form so, divide by t to arrive at $x'' - \frac{t+1}{t}x' + \frac{1}{t}x = 0$. Then, $x_2(t) = e^t \int \frac{e^{-\int -\frac{t+1}{t}}}{e^{2t}} dt = e^t \int \frac{te^t}{e^{2t}} dt = e^t \int \frac{t}{e^t} = e^t(-e^{-t}(t+1)) = -(t+1)$.

Problem 48: Use the formula for reduction of order on page 198. In order to use this formula, the equation must be in standard form so, divide by t to arrive at $y'' + \frac{1-2t}{t}y' + \frac{t-1}{t}y = 0$. Then, $y_2(t) = e^t \int \frac{e^{-\int \frac{1-2t}{t}}}{e^{2t}} dt = e^t \int \frac{e^{2t}}{te^{2t}} dt = e^t \int \frac{1}{t} = e^t \ln t$.

Section 7.2

Problem 2: $\mathscr{L}\{t^2\} = \int_0^\infty e^{-st}t^2 dt$, which can be computed using integration by parts (remember to treat s as a constant). Let $u = t^2$, then $du = 2t\,dt$ and $dv = e^{-st}$ so that $v = -\frac{e^{-st}}{s}$. Therefore, $\int e^{-st}t^2\,dt = -\frac{e^{-st}t^2}{s} + \frac{2}{s}\int e^{-st}t\,dt$. The second integral can be computed using integration by parts too. Let $u = t$, then $du = 2\,dt$ and when $dv = e^{-st}$, $v = -\frac{e^{-st}}{s}$. Then, $\int e^{-st}t = -\frac{te^{-st}}{s} + \frac{1}{s}\int e^{-st}\,dt = -\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st}$. Thus, the original integral becomes $\int e^{-st}t^2\,dt = -\frac{e^{-st}t^2}{s} + \frac{2}{s}[-\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st}] = -\frac{e^{-st}t^2}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}$. Then, the problem is to evaluate as $t \to \infty$ and at $t = 0$. As $t \to \infty$, use l'Hospital's rule and you will find this expression tends to 0 as long as $s > 0$.. At $t = 0$, the solution is $-\frac{2}{s^3}$. Therefore, the solution is $\mathscr{L}\{t^2\} = 0 - (-\frac{2}{s^3}) = \frac{2}{s^3}$ for $s > 0$.

Problem 3: $\mathscr{L}\{e^{6t}\} = \int_0^\infty e^{-st}e^{6t}dt = \int_0^\infty e^{-st+6t}dt$, which can be computed using a substitution. Let $u = -st + 6t$, then $du = (-s+6)\,dt$. Thus, $\int e^{-st+6t}\,dt = \frac{1}{-s+6}\int e^u\,du$. Therefore, the solution to the integral is $\frac{e^{-st+6t}}{-s+6}$. Then, the problem is to evaluate as $t \to \infty$ and at $t = 0$. As $t \to \infty$, then e^{-st+6t} tends to 0 as long as $s > 6$. At $t = 0$, the solution is $\frac{1}{6-s}$. Therefore, the solution is $\mathscr{L}\{t^2\} = 0 - (\frac{1}{6-s}) = \frac{1}{s-6}$ for $s > 6$.

Problem 12: $\mathscr{L}\{f(t)\} = \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st}dt$. The first integral can be computed using a substitution as in problem 3. Let $u = -st + 2t$, then $du = (-s+2)\,dt$. Thus, $\int e^{-st+2t}\,dt = \frac{1}{-s+2}\int e^u\,du$. Therefore, the solution to the first integral is $\frac{e^{-st+2t}}{-s+2}$. Evaluating this at $t = 3$ and $t = 0$ results in the solution $\frac{e^{-3s+6}-1}{2-s}$, which exists for $s > 2$. The second integral is simply $-\frac{1}{s}e^{-st}$. As $t \to \infty$, then e^{-st} tends to 0 as long as $s > 0$. At $t = 3$, the solution is $-\frac{e^{-3s}}{s}$ so that the integral evaluated at the bounds 0 to infinity is $0 - (-\frac{1}{s}) = \frac{e^{-3s}}{s}$. Therefore, the solution is $\mathscr{L}\{f(t)\} = \frac{e^{-3s+6}-1}{2-s} + \frac{e^{-3s}}{s}$ for $s > 2$.

Problem 13: Using the linearity of the Laplace transform: $\mathscr{L}\{6e^{-3t} - t^2 + 2t - 8\} = 6\mathscr{L}\{e^{-3t}\} - \mathscr{L}\{t^2\} + 2\mathscr{L}\{t\} - 8\mathscr{L}\{1\}$. Using the table of Laplace transforms, we find that this expression is equal to $6\left(\frac{1}{s+3}\right) - \frac{2}{s^3} + 2\left(\frac{1}{s^2}\right) - 8\left(\frac{1}{s}\right)$, which is defined for $s > 0$.

Problem 17: Using the linearity of the Laplace transform: $\mathscr{L}\{e^{3t}\sin 6t - t^3 + e^t\} = \mathscr{L}\{e^{3t}\sin 6t\} - \mathscr{L}\{t^3\} + \mathscr{L}\{e^t\}$. Using the table of Laplace transforms, we find that this expression is equal to $\frac{6}{(s-3)^2+36} - \frac{6}{s^4} + \frac{1}{s-1}$, which is defined for $s > 3$.

Problem 30: Each of the Laplace transform's listed in the table is a proper rational function where the degree of the denominator is greater than the degree of the numerator. Hence, as $s \to \infty$, each function approahces 0.