1. General

   1.1. Two street vendors, Nick and Ed, are selling apples. Each has exactly 30 apples to sell. Nick sells 2 apples for \$1, Ed sells 3 apples for \$2, so that if they sell all their apples Nick will get \$15, and Ed \$20. The vendors decided to put all apples in one basket, sell 5 apples for \$3, and at the end to divide the money–\$15 to Nick, \$20 to Ed. After all apples are gone the vendors' profit is \$36. Where the extra \$1 came from?

   1.2. Max and Dani are cycling between Baltimore, MD and Washington, DC. The Max is cycling from from Baltimore, MD to Washington, DC. His speed is 12 m/h. Dani is cycling from Washington, DC to Baltimore, MD. His speed is 8 m/h. The moment Max starts to move a small fly takes off his nose and flies toward Dani. Once the fly reaches Dani it turns back and flies toward Max. The fly is in the air until the cyclists meet. The fly's speed is 15 m/h. How much distance did the fly cover until the cyclists met (according to Google Maps the distance between Baltimore and Washington is 40 miles).

2. Mathematical Induction

   2.1. Show that $1^2 + 2^2 + \ldots + n^2 = \dfrac{(2n+1)(n+1)n}{6}$.

   2.2. Show that $1^3 + 2^3 + \ldots + n^3 = \left(\dfrac{(n+1)n}{2}\right)^2$.

   2.3. Note that

$$\begin{array}{lllllll} 1 & = & 1^2 & & & & \\ 1 & + & 3 & = & 2^2 & & \\ 1 & + & 3 & + & 5 & = & 3^2 \end{array}$$

   Show that $1 + 3 + 5 + \ldots + 2n - 1 = n^2$.

2.4. Consider the sum $1+x+x^2+\ldots+x^n$. If $x=1$, then

$$1+x+x^2+\ldots+x^n = 1+1^2+\ldots+1^n = \underbrace{1+1+\ldots+1}_{n+1 \text{ times}} = n+1.$$

Show that if $x \neq 1$, then $1+x+x^2+\ldots+s^n = \dfrac{x^{n+1}-1}{x-1}$.

2.5. Show that $1\times2+2\times3+\ldots+(n-3)\times(n-2) = \dfrac{(n-3)(n-2)(n-1)}{3}$.

2.6. Show that

$$(a+b)^n = \frac{n!}{(n-0)!0!}a^n b^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \ldots + \frac{n!}{(n-n)!n!}a^{n-n}b^n.$$

3. Division

3.1. **solved on February 23, 2017** by Benjamin Dunlap

if n is even, then n^2 is even. if n^2 is even, then n is even.

3.2. $\sqrt{2}$ is not a ratio of two integers.

3.3. if sum of the digits of an integer is divisible by 3, the integer is divisible by 3.

3.4. Let ab be a two digit integer. The integer $ab-ba$ is divisible by 3.

4. Inequalities

4.1. Show that for all non negative numbers x and y one has

$$\sqrt{x}+\sqrt{y} \leq \sqrt{2}\sqrt{x+y}.$$

4.2. Show that for all non negative numbers x, y, and z one has

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \leq \sqrt{3}\sqrt{x+y+z}.$$

4.3. Can you generalize Problem 4.2 to the case of n non negative numbers?

4.4. Show that for all positive numbers x, y, and z one has

$$\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} + \sqrt{\frac{z+x}{x+y+z}} \le \sqrt{6}.$$

4.5. If $x > 0$, then $x + \dfrac{1}{x} \ge 2$.

5. Geometry

5.1. A 20 ft tall tree is 50 ft away from a 25 ft tall tree. A little bird is sitting on the top of the shorter tree. The bird need to fly to the top of the taller tree, but can not do it without landing. Determine a landing point so that the flight trajectory is shortest possible.

5.2. Given n points in the plane determine the maximal number of straight lines crossing pairs of these points.