Name:

Show all work legibly.

1. (20) Solve $\left(x^2\right)^{\frac{1}{2}} = 2$.

   Solution.
   $$2 = \left(x^2\right)^{\frac{1}{2}} = \left|x^2\right|^{\frac{1}{2}} = |x|.$$

   $x = \pm 2$

2. (20) Solve $2x^4 - 6x^3 + 6x^2 - 2x = 0$.

   Solution.
   $$0 = 2x^4 - 6x^3 + 6x^2 - 2x = 2x(x-1)^3.$$

   $x = 0$ or $x = 1$.

3. (20) Consider a system of two linear equations
   $$\begin{cases} ax + y = a^2 \\ x + ay = 1 \end{cases}$$

   Assuming that a is known and $a \neq \pm 1, 0$ find x and y.

   Solution.
   $$\begin{array}{ccccc} ax & + & y & = & a^2 \\ x & + & ay & = & 1 \end{array} \rightarrow \begin{array}{ccccc} ax & + & y & = & a^2 \\ ax & + & a^2 y & = & a \end{array} \rightarrow y - a^2 y = a^2 - a \rightarrow y = -\frac{a}{1+a}.$$

   Substitution $y = -\dfrac{a}{1+a}$ into the the second equation leads to
   $$1 = x + a\left(-\frac{a}{1+a}\right) \text{ and } x = 1 + \frac{a^2}{1+a}$$

   Finally $x = 1 + \dfrac{a^2}{1+a}$, and $y = -\dfrac{a}{1+a}$.

4. • (20) Solve $1 + \dfrac{1}{1 + \dfrac{1}{x}} = x$

   Solution. Assuming that x is a solution we get
   $$1 + \frac{1}{x} = \frac{1}{x-1}, \text{ and } 2x - x^2 = x - 1.$$

   $x = \dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}$.

- (20) Solve $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}} = x$

  Solution.

  $$1 + \frac{1}{1+\dfrac{1}{x}} = \frac{1}{x} \rightarrow \frac{1}{1+\dfrac{1}{x}} = \frac{1}{x} - 1 = \frac{1-x}{x} \rightarrow 1 + \frac{1}{x} = \frac{x}{1-x} \text{ and } \frac{1}{x} = \frac{x}{1-x} - 1.$$

  Finally $2x^2 = 1$.

  $x = \pm \dfrac{1}{\sqrt{2}}.$

5. (20) True or False? $3^{\frac{1}{3}} < 4^{\frac{1}{4}}$.

   Solution.

   $$
   \begin{aligned}
   3^2 = 9 \ &> \ 8 = 2^3 \\
   3^{2/3} \ &= \ 2 \\
   3^{1/3} \ &> \ 2^{1/2} = \left(4^{1/2}\right)^{1/2} = 4^{1/4}.
   \end{aligned}
   $$

   Mark one and explain.
   ▢ True   ▢ False

6. (20) Let x and y be non negative numbers. True or False? $\sqrt{xy} > \dfrac{x+y}{2}$

   Solution.

   $$
   \begin{aligned}
   0 \ &\le \ (x-y)^2 = x^2 - 2xy + y^2 \\
   4xy \ &\le \ x^2 + 2xy + y^2 \\
   xy \ &\le \ \frac{x^2 + 2xy + y^2}{4} = \frac{(x+y)^2}{4} \\
   \sqrt{xy} \ &\le \ \frac{x+y}{2}
   \end{aligned}
   $$

   Mark one and explain.
   ▢ True   ▢ False

7. (20) True or False? $1 < \dfrac{1}{101} + \dfrac{1}{102} + \dfrac{1}{103} + \ldots + \dfrac{1}{200}$

   Solution.

   $$\frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \ldots + \frac{1}{200} \le \underbrace{\frac{1}{100} + \ldots + \frac{1}{100}}_{100 \text{ times}} = 1.$$

   Mark one and explain.
   ▢ True   ▢ False

8. (20) If x and y are non negative numbers and $x + y = 1$. What is the maximal possible value for $x - y$? $x^2 - y^2$?

Solution.

(a) Let $x \geq y$. Since $1 \geq x$ and $y > 0$ one has $1 \geq x - y \geq 0$. If $x = 1$ and $y = 0$, then $1 = x - y$.

(b) $x^2 - y^2 = (x - y)(x + y) = x - y$, we already know that the maximal possible value for $x - y$ is 1.