MATH 225, SPRING 2017 - HOMEWORK #2

Due Thursday, February 16

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Section 2.4

Problem 10: For the equation $(2xy+3)dx+(x^2-1)dy=0$ let $M(x,y)=2xy+3$ and $N(x,y)=x^2-1$. Then, $\frac{\partial M}{\partial y}=2x$ and $\frac{\partial N}{\partial x}=2x$, so the equation is exact. To solve the equation, first integrate M with respect to x so that $F(x,y)=\int(2xy+3)dx+g(y)=x^2y+3x+g(y)$. Then, take the partial derivative of F with respect to $y$and set that equal to N: $\frac{\partial F}{\partial y}=x^2+g'(y)=x^2-1=N(x,y)$. Finaly, solve for $g'(y)$: $g'(y)=-1$ so that $g(y)=-y$. Therefore, $F(x,y)=x^2y+3x-y$ and $x^2y+3x-y=C$ is a general solution.

Problem 15: For the equation $\cos\theta dr-(r\sin\theta-e^\theta)d\theta=0$ let $M(r,\theta)=\cos\theta$ and $N(r,\theta)=-(r\sin\theta-e^\theta)$. Then, $\frac{\partial M}{\partial\theta}=-\sin\theta$ and $\frac{\partial N}{\partial r}=-\sin\theta$, so the equation is exact. To solve the equation, first integrate M with respect to r so that $F(r,\theta)=\int(\cos\theta)dr+g(\theta)=r\cos\theta+g(\theta)$. Then, take the partial derivative of F with respect to θ and set that equal to N: $\frac{\partial F}{\partial\theta}=-r\sin\theta+g'(\theta)=-(r\sin\theta-e^\theta)=N(r,\theta)$. Finally, solve for $g'(\theta)$: $g'(\theta)=e^\theta$ so that $g(\theta)=e^\theta$. Therefore, $F(r,\theta)=r\cos\theta+e^\theta$ and $r\cos\theta+e^\theta=C$ is a general solution.

Problem 21: For the equation $(1/x+2y^2x)dx+(2yx^2-\cos y)dy=0$ let $M(x,y)=1/x+2y^2x$ and $N(x,y)=2yx^2-\cos y$. Then, $\frac{\partial M}{\partial y}=4yx$ and $\frac{\partial N}{\partial x}=4yx$, so the equation is exact. To solve the equation, first integrate M with respect to x so that $F(x,y)=\int(1/x+2y^2x)dx+g(y)=\ln|x|+y^2x^2+g(y)$. Then, take the partial derivative of F with respect to y and set that equal to N: $\frac{\partial F}{\partial y}=2yx^2+g'(y)=2yx^2-\cos y=N(x,y)$. Finaly, solve for $g'(y)$: $g'(y)=-\cos y$ so that $g(y)=-\sin y$. Therefore, $F(x,y)=\ln|x|+y^2x^2-\sin y$ and $\ln|x|+y^2x^2-\sin y=C$ is a general solution. Using the initial condition $y(1)=\pi$, let $y=\pi$ and $x=1$ and solve for C: $C=\ln 1+\pi^2-\sin\pi=\pi^2$, so that the solution of the equation is $\ln|x|+y^2x^2-\sin y=\pi^2$.

Problem 25: For the equation $(y^2\sin x)dx+(1/x-y/x)dy=0$ let $M(x,y)=y^2\sin x$ and $N(x,y)=1/x-y/x$. Then, $\frac{\partial M}{\partial y}=2y\sin x$ and $\frac{\partial N}{\partial x}=\frac{y}{x^2}-\frac{1}{x^2}$, so the equation is not exact. However, the equation is separable: $(y^2\sin x)dx+(1/x-y/x)dy=0\Rightarrow y^2\sin xdx=-\frac{1-y}{x}dy\Rightarrow x\sin xdx=\frac{y-1}{y^2}dy$. Integrating both sides yields $\sin x-x\cos x+C=\ln|y|+\frac{1}{y}$. Use the initial condition $y(\pi)=1$ and solve for C by letting $x=\pi$ and $y=1$: $\sin(\pi)-\pi\cos(\pi)+C=\ln(1)+1\Rightarrow\pi+C=1\Rightarrow C=1-\pi$. Therefore, the solution to the equation is $\sin x-x\cos x+1-\pi=\ln|y|+\frac{1}{y}$ or $\sin x-x\cos x=\ln|y|+\frac{1}{y}+\pi-1$.

Problem 27b: For the equation $M(x,y)dx+(\sin x\cos y-xy-e^{-y})dy=0$, if $N(x,y)=\sin x\cos y-xy-e^{-y}$, we need $M(x,y)$ so that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=\cos x\cos y-y$. Therefore, $M(x,y)=\int(\cos x\cos y-y)dy=\cos x\sin y-\frac{y^2}{2}+f(x)$. Check: $\frac{\partial M}{\partial y}=\cos x\cos y-y=\frac{\partial N}{\partial x}$.

Problem 33a: For this problem, we need to solve the equation $\frac{\partial F}{\partial y}(x,y)dx-\frac{\partial F}{\partial x}(x,y)dy=0$. For the equation $2x^2+y^2=k$, $\frac{\partial F}{\partial y}=2y$ and $\frac{\partial F}{\partial x}=4x$ so the differential equation is $2ydx-4xdy=0$, which is separable: $\frac{dx}{x}=2\frac{dy}{y}\Rightarrow\ln|x|+C=2\ln y\Rightarrow y^2=e^{\ln x+C}\Rightarrow y^2=Cx$, or $x=Cy^2$. An additional solution is $x=0$, $y=0$.

Section 2.6

Problem 1: The equation $2txdx+(t^2-x^2)dt=0$ can be rewritten as $\frac{dx}{dt}=\frac{x^2-t^2}{2tx}=\frac{1}{2}\left(\frac{x}{t}-\frac{t}{x}\right)$, which is a function of $\frac{x}{t}$, so the equation is homogeneous. Notice that the function can also be written as $\frac{dx}{dt}-\frac{1}{2t}x=-\frac{t}{2}x^{-1}$, so the equation is also a Bernoulli equation.

Problem 11: The equation $(y^2 - xy)dx + x^2dy = 0$ can be rewritten as $\frac{dy}{dx} = \frac{xy-y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$. Let $v = \frac{y}{x}$. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx} = v - v^2 \Rightarrow -\frac{dv}{v^2} = \frac{dx}{x}$. So, $\frac{1}{v} = \ln|x| + C$. Substituting $v = \frac{y}{x}$, we get $\frac{x}{y} = \ln|x| + C$ or $y = \frac{x}{\ln|x|+C}$. An additional solution is $x = 0$, $y = 0$.

Problem 13: The equation $\frac{dx}{dt} = \frac{x^2+t\sqrt{t^2+x^2}}{tx}$ can be rewritten as $\frac{dx}{dt} = \frac{x}{t} + \frac{\sqrt{t^2+x^2}}{x} = \frac{x}{t} + \frac{\sqrt{1+(\frac{x}{t})^2}}{\frac{x}{t}}$. Let $v = \frac{x}{t}$. Then , $\frac{dx}{dt} = v + t\frac{dv}{dt} = v + \frac{\sqrt{1+v^2}}{v}$. Thereore, $\frac{v}{\sqrt{1+v^2}}dv = \frac{dt}{t}$. The integral $\int \frac{v}{\sqrt{1+v^2}}dv$ can be computed using substitution, letting $u = 1 + v^2$ so $du = 2vdv$, then the integral is $\frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{1+v^2}$. Therefore, the solution is $\sqrt{1+v^2} = \ln|t| + C$. Substituting $v = \frac{x}{t}$, we get $\sqrt{1+\left(\frac{x}{t}\right)^2} = \ln|t| + C$.