MATH 225, SPRING 2017 - HOMEWORK #6

Due Thursday, April 13

Section 7.2, page 360: 2, 3, 12, 13, 17, 30
Section 7.3, page 365: 4, 9, 21, 22, 27, 31
Section 7.4, page 374: 4, 10, 22, 27, 31

Section 7.2

Problem 2: $\mathscr{L}\{t^2\} = \int_0^\infty e^{-st}t^2 dt$, which can be computed using integration by parts (remember to treat s as a constant). Let $u = t^2$, then $du = 2t\,dt$ and $dv = e^{-st}$ so that $v = -\frac{e^{-st}}{s}$. Therefore, $\int e^{-st}t^2\,dt = -\frac{e^{-st}t^2}{s} + \frac{2}{s}\int e^{-st}t\,dt$. The second integral can be computed using integration by parts too. Let $u = t$, then $du = 2\,dt$ and when $dv = e^{-st}$, $v = -\frac{e^{-st}}{s}$. Then, $\int e^{-st}t = -\frac{te^{-st}}{s} + \frac{1}{s}\int e^{-st}\,dt = -\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st}$. Thus, the original integral becomes $\int e^{-st}t^2\,dt = -\frac{e^{-st}t^2}{s} + \frac{2}{s}[-\frac{te^{-st}}{s} - \frac{1}{s^2}e^{-st}] = -\frac{e^{-st}t^2}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}$. Then, the problem is to evaluate as $t \to \infty$ and at $t = 0$. As $t \to \infty$, use l'Hospital's rule and you will find this expression tends to 0 as long as $s > 0$.. At $t = 0$, the solution is $-\frac{2}{s^3}$. Therefore, the solution is $\mathscr{L}\{t^2\} = 0 - (-\frac{2}{s^3}) = \frac{2}{s^3}$ for $s > 0$.

Problem 3: $\mathscr{L}\{e^{6t}\} = \int_0^\infty e^{-st}e^{6t}dt = \int_0^\infty e^{-st+6t}dt$, which can be computed using a substitution. Let $u = -st + 6t$, then $du = (-s+6)\,dt$. Thus, $\int e^{-st+6t}\,dt = \frac{1}{-s+6}\int e^u\,du$. Therefore, the solution to the integral is $\frac{e^{-st+6t}}{-s+6}$. Then, the problem is to evaluate as $t \to \infty$ and at $t = 0$. As $t \to \infty$, then e^{-st+6t} tends to 0 as long as $s > 6$. At $t = 0$, the solution is $\frac{1}{6-s}$. Therefore, the solution is $\mathscr{L}\{t^2\} = 0 - (\frac{1}{6-s}) = \frac{1}{s-6}$ for $s > 6$.

Problem 12: $\mathscr{L}\{f(t)\} = \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st}dt$. The first integral can be computed using a substitution as in problem 3. Let $u = -st + 2t$, then $du = (-s+2)\,dt$. Thus, $\int e^{-st+2t}\,dt = \frac{1}{-s+2}\int e^u\,du$. Therefore, the solution to the first integral is $\frac{e^{-st+2t}}{-s+2}$. Evaluating this at $t = 3$ and $t = 0$ results in the solution $\frac{e^{-3s+6}-1}{2-s}$, which exists for $s > 2$. The second integral is simply $-\frac{1}{s}e^{-st}$. As $t \to \infty$, then e^{-st} tends to 0 as long as $s > 0$. At $t = 3$, the solution is $-\frac{e^{-3s}}{s}$ so that the integral evaluated at the bounds 0 to infinity is $0 - (-\frac{1}{s}) = \frac{e^{-3s}}{s}$. Therefore, the solution is $\mathscr{L}\{f(t)\} = \frac{e^{-3s+6}-1}{2-s} + \frac{e^{-3s}}{s}$ for $s > 2$.

Problem 13: Using the linearity of the Laplace transform: $\mathscr{L}\{6e^{-3t} - t^2 + 2t - 8\} = 6\mathscr{L}\{e^{-3t}\} - \mathscr{L}\{t^2\} + 2\mathscr{L}\{t\} - 8\mathscr{L}\{1\}$. Using the table of Laplace transforms, we find that this expression is equal to $6\left(\frac{1}{s+3}\right) - \frac{2}{s^3} + 2\left(\frac{1}{s^2}\right) - 8\left(\frac{1}{s}\right)$, which is defined for $s > 0$.

Problem 17: Using the linearity of the Laplace transform: $\mathscr{L}\{e^{3t}\sin 6t - t^3 + e^t\} = \mathscr{L}\{e^{3t}\sin 6t\} - \mathscr{L}\{t^3\} + \mathscr{L}\{e^t\}$. Using the table of Laplace transforms, we find that this expression is equal to $\frac{6}{(s-3)^2+36} - \frac{6}{s^4} + \frac{1}{s-1}$, which is defined for $s > 3$.

Problem 30: Each of the Laplace transform's listed in the table is a proper rational function where the degree of the denominator is greater than the degree of the numerator. Hence, as $s \to \infty$, each function approahces 0.

Section 7.3

Problem 4: Using linearity $\mathscr{L}\{3t^4 - 2t^2 + 1\} = 3\mathscr{L}\{t^4\} - 2\mathscr{L}\{t^2\} + \mathscr{L}\{1\} = 3\left(\frac{4!}{s^5}\right) - 2\left(\frac{2!}{s^3}\right) + \frac{1}{s} = \frac{72}{s^5} - \frac{4}{s^3} + \frac{1}{s}$, which is defined for $s > 0$.

Problem 9: Using Table 7.2, $\mathscr{L}\{e^{-t}t\sin(2t)\} = \mathscr{L}\{te^{-t}\sin(2t)\} = (-1)^1\frac{d}{ds}(\mathscr{L}\{e^{-t}\sin(2t)\}$. Using Table 7.1, $\mathscr{L}\{e^{-t}\sin(2t)\} = \frac{2}{(s+1)^2+4}$. The derivatvie of this function is $\frac{-4(s+1))}{((s+1)^2+4)^2}$. Therefore, $\mathscr{L}\{e^{-t}t\sin(2t)\} = (-1)^1\frac{d}{ds}(\mathscr{L}\{e^{-t}\sin(2t)\} = -1\left(\frac{-4(s+1))}{((s+1)^2+4)^2}\right) = \frac{4(s+1)}{((s+1)^2+4)^2}$.

Problem 21: Using the translation property, $\mathscr{L}\{e^{at}\cos bt\} = \mathscr{L}\{\cos bt\}(s-a) = \frac{s-a}{(s-a)^2+b^2}$. Note: $\mathscr{L}\{\cos bt\}(s-a)$ is the Laplace transform of $\cos bt$ (which is a function of s) evaluated at $(s-a)$, not multiplied!

Problem 22: We know when $n = 0$, $\mathscr{L}\{t^0\} = \mathscr{L}\{1\} = \frac{1}{s}$. Using formula 6, $\mathscr{L}\{t^n\} = \mathscr{L}\{t^n \cdot 1\} = (-1)^n \cdot \frac{d^n}{ds^n}(\mathscr{L}\{1\}) = (-1)^n \frac{d^n}{ds^n}(\frac{1}{s})$. At this point, notice that $\frac{d^n}{ds^n}(\frac{1}{s}) = (-1)(-2)\cdots(-n)\frac{1}{s^{n+1}} = (-1)^n(1\cdot 2\cdots n)\frac{1}{s^{n+1}} = (-1)^n(n!)\frac{1}{s^{n+1}}$. Therefore, $(-1)^n \frac{d^n}{ds^n}(\frac{1}{s}) = (-1)^n(-1)^n(n!)\frac{1}{s^{n+1}} = (-1)^{2n}(n!)\frac{1}{s^{n+1}}$. However, note that $2n$ is always an even integer; which means $(-1)^{2n}$ is always equal to 1. Therefore, we get $\frac{n!}{s^{n+1}}$ as the final solution.

Problem 27: Use the hint. Since f is piecewise continuous and of exponential order its Laplace transform exists. Let $F(s)$ be the Laplace transform of f. Then, by definition, $F(s) = \int_0^\infty e^{-st}f(t)\,dt$. Then, integrate both sides with respect to s, $\int_s^\infty F(s)\,ds = \int_s^\infty[\int_0^\infty e^{-st}f(t)\,dt]\,ds$. Then, by Leibniz's rule we interchange the order of integration so that $\int_s^\infty F(s)\,ds = \int_0^\infty[\int_s^\infty e^{-st}f(t)\,ds]\,dt = \int_0^\infty f(t)[\lim_{a\to\infty}\frac{-e^{-st}}{t}]_s^\infty\,dt = \int_0^\infty f(t)\frac{e^{-st}}{t}\,dt = \mathscr{L}\{\frac{f(t)}{t}\}$. Therefore, $\frac{d}{ds}\mathscr{L}\{\frac{f(t)}{t}\} = \frac{d}{ds}\int_s^\infty F(s)\,ds = \lim_{a\to\infty} F(a) - F(s) = 0 - F(s) = -F(s)$ (using the result of part (b) in Problem 26 because $F(s) = \mathscr{L}\{f\}(s)$. So, $F(s) = \int_s^\infty F(u)\,du$.

Problem 31: $\mathscr{L}\{g(t)\} = \int_0^c e^{-st}0\,dt + \int_c^\infty e^{-st}f(t-c)\,dt$. The first integral is 0. For the second integral, let $t = x + c$, so that it becomes $\int_0^\infty f(x)e^{-s(x+c)}\,dx = \int_0^\infty f(x)e^{-sx}e^{-sc}dx = e^{-sc}\int_0^\infty e^{-sx}f(x)\,dx = e^{-sc}\mathscr{L}\{f\}(s)$.

Section 7.4

Problem 4: $\mathscr{L}^{-1}\{\frac{4}{s^2+9}\}(t) = \mathscr{L}^{-1}\{\frac{\frac{4}{3}\cdot 3}{s^2+3^2}\}(t) = \frac{4}{3}\mathscr{L}^{-1}\{\frac{3}{s^2+3^2}\}(t) = \frac{4}{3}\sin 3t$

Problem 10: $\mathscr{L}^{-1}\{\frac{s-1}{2s^2+s+6}\}(t) = \mathscr{L}^{-1}\{\frac{s-1}{2(s^2+s/2+3)}\}(t) = \frac{1}{2}\mathscr{L}^{-1}\{\frac{s-1}{(s+1/4)^2+47/16}\}(t)$. So, $\frac{1}{2}\mathscr{L}^{-1}\{\frac{s+1/4}{(s+1/4)^2+(\sqrt{47}/4)^2} - \frac{5}{\sqrt{47}}\frac{\sqrt{47}/4}{(s+1/4)^2+(\sqrt{47}/4)^2}\}(t) = \frac{1}{2}e^{-t/4}\cos(\frac{\sqrt{47}t}{4}) - \frac{5}{2\sqrt{47}}e^{-t/4}\sin(\frac{\sqrt{47}t}{4})$

Problem 22: To compute $\mathscr{L}^{-1}\{\frac{s+11}{(s-1)(s+3)}\}(t)$, first use partial fractions to decompose the rational function $\frac{s+11}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} \Rightarrow As + 3A + Bs - B = s + 11$. Therefore, $A + B = 1$ and $3A - B = 11$. Solving this system yields $A = 3$, $B = -2$. So, we get $\mathscr{L}^{-1}\{\frac{s+11}{(s-1)(s+3)}\}(t) = \mathscr{L}^{-1}\{\frac{3}{s-1} - \frac{2}{s+3}\}(t) = 3e^t - 2e^{-3t}$.

Problem 27: In order to do this problem, first solve for $F(s)$:$F(s)[s^2-4] = \frac{5}{s+1} \Rightarrow F(s) = \frac{5}{s+1}\cdot\frac{1}{s^2-4} = \frac{5}{(s+1)(s+2)(s-2)}$. Then, use partial fractions to decmopose the rational function $\frac{5}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2} \Rightarrow A(s^2-4) + B(s^2-s-2) + C(s^2+3s+2) = 5$. Therefore, $A + B + C = 0$, $-B + 3C = 0$, and $-4A - 2B + 2C = 5$. Solving this system yields $A = -\frac{5}{3}$, $B = \frac{5}{4}$, $C = \frac{5}{12}$. Then, we are solving $= \mathscr{L}^{-1}\{\frac{-5}{3(s+1)} + \frac{5}{4(s+2)} + \frac{5}{12(s-2)}\}(t) = -\frac{5}{3}e^{-t} + \frac{5}{4}e^{-2t} + \frac{5}{12}e^{2t}$.

Problem 31: The Laplace transform, by definition, is a definite integral. So, the discrete points at which the function is defined do not matter. Therefore, $\mathscr{L}\{f_1\} = \mathscr{L}\{t\} = \frac{1}{s^2}$, $\mathscr{L}\{f_2\} = \mathscr{L}\{t\} = \frac{1}{s^2}$, $\mathscr{L}\{f_3\} = \mathscr{L}\{t\} = \frac{1}{s^2}$. By definition of the inverse Laplace transform, the function $f = \mathscr{L}^{-1}\{F(s)\}$ must be continuous. Since f_1 and f_2 have points of discontinuity, the inverse Laplace transform of $1/s^2$ is $f(t) = t$.