MATH225
quiz #3, 04/25/17
Total 100
Solutions

Show all work legibly. **Name**:______________________

1. (10) Let $g(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$. Express $g(t)$ as a combination of step functions.

   The solution is $g(t) =$

   Solution. $g(t) = u(t) - u_\pi(t)$.

2. (15) Let $g(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$. Find $G(s)$, the Laplace transform of $g(t)$.

   The solution is $G(s) =$

   Solution. Since $g(t) = u(t) - u_\pi(t)$, one has $G(s) = \frac{1}{s}\left[1 - e^{-\pi s}\right]$.

3. (30) Find the inverse Laplace transform of $\frac{1}{s(s^2+s+\frac{5}{4})}$.

   The inverse Laplace transform is:

   Solution. $\frac{1}{s(s^2+s+\frac{5}{4})} = \frac{a}{s} + \frac{bs+c}{s^2+s+\frac{5}{4}}$. Multiplication of both sides by $s\left(s^2+s+\frac{5}{4}\right)$ leads to

$$1 = a\left(s^2+s+\frac{5}{4}\right) + (bs+c)s.$$

If $s = 0$, then $a = \frac{4}{5}$. Differentiation of the identity $1 = \frac{4}{5}\left(s^2+s+\frac{5}{4}\right) + (bs+c)s$ leads to

$$0 = \frac{4}{5}(2s+1) + (2bs+c).$$

and when $s = 0$ one has $c = -\frac{4}{5}$. The identity then becomes

$$0 = \frac{4}{5}(2s+1) + 2bs - \frac{4}{5}.$$

Equating coefficients of like powers of s one has $b = -\frac{4}{5}$. Finally

$$\frac{1}{s(s^2+s+\frac{5}{4})} = \frac{4}{5}\frac{1}{s} - \frac{4}{5}\frac{s+1}{s^2+s+\frac{5}{4}} = \frac{4}{5}\frac{1}{s} - \frac{4}{5}\frac{\left[s+\frac{1}{2}\right]+\frac{1}{2}}{\left[s+\frac{1}{2}\right]^2+1}.$$

The inverse Laplace transform of this function is

$$\frac{4}{5}-\frac{4}{5}\left[e^{-\frac{t}{2}}\cos\ t+\frac{1}{2}e^{-\frac{t}{2}}\sin t\right].$$

4. (30) Find the solution $y(t)$ of the differential equation

$$y''+y'+\frac{5}{4}y=g(t)=\begin{cases}1 & 0\le t<\pi\\ 0 & t\ge\pi\end{cases}$$

with the initial conditions $y(0)=0,\ y'(0)=0$.

The solution is $y(t)=$

Solution. Application of the Laplace transform to both sides of the equation leads to

$$Y=\frac{1-e^{-\pi s}}{s(s^2+s+\frac{5}{4})}.$$

First we find the inverse Laplace transform of $\frac{1}{s(s^2+s+\frac{5}{4})}$. Note that

$$\frac{1}{s(s^2+s+\frac{5}{4})}=\frac{a}{s}+\frac{bs+c}{s^2+s+\frac{5}{4}}\text{ with }a=\frac{4}{5},\text{ and }b=c=-\frac{4}{5}.$$

That is

$$\frac{1}{s(s^2+s+\frac{5}{4})}=\frac{4}{5}\frac{1}{s}-\frac{4}{5}\frac{[s+\frac{1}{2}]+\frac{1}{2}}{\left[s+\frac{1}{2}\right]^2+1}$$

and the inverse Laplace transform of this function is

$$\frac{4}{5}-\frac{4}{5}\left[e^{-\frac{t}{2}}\cos\ t+\frac{1}{2}e^{-\frac{t}{2}}\sin t\right].$$

The inverse Laplace transform of $e^{-\pi s}\frac{1}{s(s^2+s+\frac{5}{4})}$ is

$$u_\pi(t)\left(\frac{4}{5}-\frac{4}{5}\left[e^{-\frac{t}{2}-\pi}\cos\ (t-\pi)+\frac{1}{2}e^{-\frac{t}{2}-\pi}\sin(t-\pi)\right]\right).$$

Finally

$$y(t)=\frac{4}{5}-\frac{4}{5}\left[e^{-\frac{t}{2}}\cos\ t+\frac{1}{2}e^{-\frac{t}{2}}\sin t\right]-u_\pi(t)\left(\frac{4}{5}+\frac{4}{5}\left[e^{-\frac{t}{2}-\pi}\cos\ t+\frac{1}{2}e^{-\frac{t}{2}-\pi}\sin\ t\right]\right).$$

5. (15) Let $f(t)$ be a periodic function with period $T = 1$ and values $f(t) = t$ for $0 \le t \le 1$. Find $F(s)$ the Laplace transform of $f(t)$.

The solution is $F(s) =$

Solution. We start with evaluation of $F_1(s)$ the Laplace transform of $f_1(t) = u(t)f(t) - u(t-1)f(t-1)$.

$$\begin{aligned} F_1(s) &= \int_0^1 e^{-st} f(t)\, dt = -\frac{1}{s} e^{-st}|_0^1 + \frac{1}{s} \int_0^1 e^{-st}\, dt \\ &= -\frac{1}{s} e^{-s} + \frac{1}{s^2}(1 - e^{-s}) \end{aligned}$$

Next

$$F(s) = \frac{F_1(s)}{1 - e^{-s}} = -\frac{1}{s} \frac{e^{-s}}{1 - e^{-s}} + \frac{1}{s^2}.$$