MATH225
quiz #4, 05/09/17
Total 100
Solutions

Show all work legibly. **Name**:______________________

1. (10) Find the inverse Laplace transform $f(t)$ of $\frac{1}{s(2s^2+10)}$.

   The solution is $f(t) =$

   Solution. Note that
$$\frac{1}{s(2s^2+10)} = \frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s}{s^2+5}, \text{ and } f(t) = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}t\right).$$

2. (10) Find the inverse Laplace transform $f(t)$ of $\frac{s+2}{s^2+5}$.

   The solution is $f(t) =$

   Solution. Note that
$$\frac{s+2}{s^2+5} = \frac{s}{s^2+5} + \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{s^2+5}.$$
   Hence $f(t) = \cos\left(\sqrt{5}t\right) + \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}t\right)$.

3. (30) Solve the initial value problem $2y''+10y = 3u_{12}(t)-5\delta(t-4),\ y(0) = -1,\ y'(0) = -2$.

   The solution is $y(t) =$

   Solution. Application of the Laplace transform to both sides of the equation leads to
$$2\left(s^2Y + s + 2\right) + 10Y = 3\frac{e^{-12s}}{s} - 5e^{-4s}.$$
   and
$$Y = \frac{3e^{-12s}}{s(2s^2+10)} - \frac{5e^{-4s}}{(2s^2+10)} - \frac{2s+4}{(2s^2+10)}.$$
   Since

   (a) $L^{-1}\left(\frac{1}{s(2s^2+10)}\right) = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}t\right)$, one has
$$L^{-1}\left(\frac{3e^{-12s}}{s(2s^2+10)}\right) = 3u_{12}(t)\left[\frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}(t-12)\right)\right].$$

(b) $L^{-1}\left(\frac{1}{s^2+5}\right) = L^{-1}\left(\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+5}\right) = \frac{1}{\sqrt{5}}\sin\sqrt{5}t$ one has

$$L^{-1}\left(\frac{5e^{-4s}}{(2s^2+10)}\right) = \frac{5}{2}\frac{1}{\sqrt{5}}\sin\left(\sqrt{5}(t-4)\right)u(t-4) = \frac{\sqrt{5}}{2}\sin\left(\sqrt{5}(t-4)\right)u(t-4)$$

Finally

$$\begin{aligned} f(t) &= 3u_{12}(t)\left[\frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}(t-12)\right)\right] \\ &- \frac{\sqrt{5}}{2}\sin\left(\sqrt{5}(t-4)\right)u(t-4) \\ &- \cos\left(\sqrt{5}t\right) - \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}t\right). \end{aligned}$$

4. (10) Find the inverse Laplace transform $f(t)$ of $\frac{s}{(s^2+1)}$.

The solution is $f(t) =$

Solution. From the table one has $f(t) = \cos t$.

5. (10) Find the inverse Laplace transform $f(t)$ of $\frac{2s}{(s^2+1)^2}$.

The solution is $f(t) =$

Solution. From the table one has $f(t) = t\sin t$.

6. (30) Find $f(t)$ that solves $f(t) = 2\cos t - \int_0^t (t-\tau)f(\tau)\, d\,\tau$ (Hint: the integral is the convolution of two function. An application of the Laplace transform to the integral results in the product of Laplace transforms of these two functions.)

The solution is $f(t) =$

Solution. Let $F(s) = L(f(t))$, and $G(s) = L(t) = \frac{1}{s^2}$. Application of the Laplace transform to both sides of the relation defining $f(t)$ leads to

$$F(s) = \frac{2s}{s^2+1} - \frac{1}{s^2}F(s), \text{ and } F(s) = \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2}.$$

Note that:

(a) $L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$

(b) $L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t\sin t$

Finally $f(t) = 2\cos t - t\sin t$.