MATH221-04
quiz #2, 11/01/18
Total 100
Solutions

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Show all work legibly. **Name**:______________________

1. (20) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of linearly independent vectors. True or False? The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

   Solution. If $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set one has $c_1 = c_2 = 0$.

   Mark one and explain.
   ▫ True ▫ False

2. (20) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. Find A^{-1} if exists.

   Solution.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}.$$

3. (20) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}$. Find all 2×2 matrices X so that $AX = B$.

Solution. Let $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. If $AX = B$, then

$$AX = \left[A\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \ A\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \right] = \begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}.$$

That leads to two systems of linear equations

$$A\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \text{ and } A\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

The solutions are

$$\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = x_{21}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = x_{22}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix} + x_{21}\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + x_{22}\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}.$$

4. (20) Let A be an invertible matrix. True or False? If $AB = AC$, then $B = C$.

   Solution. If $AB = AC$, then $A^{-1}AB = A^{-1}AC$, and $B = C$.

   Mark one and explain.
   ▫ True ▫ False

5. (20) Let span $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \mathbf{R}^n$. True or False? The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbf{R}^n$.

   Solution. Let $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$. Each linear system $U\mathbf{x} = \mathbf{e}_i$ has a solution. Denote this solution by $\mathbf{a}_i$, i.e. $U\mathbf{a}_i = \mathbf{e}_i$. If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, then $UA = I$, and U is an invertible matrix. This shows that columns of U are linearly independent, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbf{R}^n$.

   Mark one and explain.
   ▫ True ▫ False

6. (20) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. True or False? If $\{[\mathbf{u}_1]_\mathcal{B}, \ldots, [\mathbf{u}_n]_\mathcal{B}\}$ is a linearly independent set, then the vector set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent.

   Solution. Let $\mathbf{u}_i = c_{1i}\mathbf{b}_1 + \ldots + c_{ni}\mathbf{b}_n$, i.e. $[\mathbf{u}_i]_\mathcal{B} = \begin{bmatrix} c_{1i} \\ \ldots \\ c_{ni} \end{bmatrix}$. Consider the equation

$$0 = x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n.$$

   Note that

$$0 = x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n = (x_1c_{11} + x_2c_{12} + \ldots + x_nc_{1n})\mathbf{b}_1 + \ldots + (x_1c_{n1} + x_2c_{n2} + \ldots + x_nc_{nn})\mathbf{b}_n.$$

Since the vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are linearly independent one has

$$\begin{aligned} x_1 c_{11} + x_2 c_{12} + \ldots + x_n c_{1n} &= 0 \\ &\ldots \\ x_1 c_{n1} + x_2 c_{n2} + \ldots + x_n c_{nn} &= 0 \end{aligned}$$

In other words if C is the matrix whose i^{th} column is $[\mathbf{u}_i]_{\mathcal{B}}$, then $C\mathbf{x} = 0$. The vectors $[\mathbf{u}_i]_{\mathcal{B}}$ are linearly independent, hence C^{-1} exists, and $\mathbf{x} = 0$.

Mark one and explain.

□ True □ False