MATH430

fall 09 review problems

1. True of False? If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ is an $n \times m$ matrix and $\forall\, \mathbf{x} \in \mathbf{R}^m$ one has $A\mathbf{x} = \mathbf{0}$, then $A = 0$.

2. True of False? If $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a linearly dependent vector set, then one of the vectors is a linear combination of the others.

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4. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a linearly independent vector set. If the the entries b_{ij} of the matrix B are defined by $b_{ij} = \mathbf{a}_i^T \mathbf{a}_j$, then $\det B \neq 0$ (i.e. rank $B = n$, and n columns of B are linearly independent).

5. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a linearly independent vector set in a vector space $\mathbf{R}^m$. If $\mathbf{x} \in \mathbf{R}^m$, and $\mathbf{x} \notin \text{span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, then $\mathbf{x} = \mathbf{a} + \mathbf{b}$ such that $\mathbf{a} \in \text{span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, and $\mathbf{b}^T \mathbf{a}_i = 0$, $i = 1, \ldots, n$.

6. If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ is an $n \times m$ matrix and $\mathbf{x} \in \mathbf{R}^m$, then $A\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_m\mathbf{a}_m$.

7. If A is an $n \times m$ matrix and $B = [\mathbf{b}_1, \ldots, \mathbf{b}_k]$, $\mathbf{b}_i \in \mathbf{R}^m$, then $AB = [A\mathbf{b}_1, \ldots, A\mathbf{b}_k]$.

8. True of False? If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$, $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, and the vestor sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are linearly independent, then the set $\{A\mathbf{b}_1, \ldots, A\mathbf{b}_n\}$ is linearly independent.

9. True of False? If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$, $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, and the vestor set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly dependent, then the set $\{A\mathbf{b}_1, \ldots, A\mathbf{b}_n\}$ is linearly dependent.

10. If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$, then $AA^T = \mathbf{a}_1\mathbf{a}_1^T + \ldots + \mathbf{a}_m\mathbf{a}_m^T$.

11. True of False? If A and P are $n \times n$ matrices so that $AP = A$, then $P = I$.

12. Let $\mathbf{u}$, $\mathbf{v} \in \mathbf{R}^n$ so that $\mathbf{v}^T\mathbf{u} \neq 1$. For the matrices $I - \mathbf{u}\mathbf{v}^T$ and $I - \dfrac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u} - 1}$ compute the product $\left(I - \mathbf{u}\mathbf{v}^T\right)\left(I - \dfrac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u} - 1}\right)$.

13. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix}$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    - For $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_2$ compute $E_1 = I - \mathbf{u}\mathbf{u}^T$ and $\left(I - \mathbf{u}\mathbf{u}^T\right)A = E_1 A$.
    - Compute $E_2 = I - (1 - \alpha)\mathbf{e}_2\mathbf{e}_2^T$, and $E_2 A$.
    - Compute $E_3 = I - \alpha\mathbf{e}_2\mathbf{e}_1^T$, and $E_3 A$.

14. Elementary row operations:

    - multiplication of a row by a nonzero number,
    - switching two different rows,

- addition of a multiple of row i to row j.

(a) Reduced row echelon form of A.

(b) If B is obtained from A by a single elementary row operation, and k rows of A are linearly independent, then there are k linearly independent rows of B.

(c) If B is obtained from A by a single elementary row operation, then $R(A^T) = R(B^T)$.

(d) If B is obtained from A by a single elementary row operation, then there is an invertible matrix E such that $EA = B$ (the matrix E is called an elementary matrix).

(e) For each A there is a finite set of elementary mtrices $E_1, E_2, \ldots, E_n$ so that $E_n E_{n-1} \ldots E_2 E_1 A$ is in a row echelon form.

(f) If B is obtained from A by a single elementary row operation, then $N(B) = N(A)$.

(g) Apply elementary row operations to an $n \times m$ matrix A and reduce it to

$$\begin{bmatrix} I_k & B \\ 0_{(n-k)\times k} & 0_{(n-k)\times(m-k)} \end{bmatrix}$$

$B = [\mathbf{b}_1, \ldots, \mathbf{b}_{m-k}]$ is $k \times (m-k)$. Let $\hat{B} = [\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_{m-k}]$, where $\hat{\mathbf{b}}_i = (\mathbf{b}_i^T, \mathbf{e}_i^T)^T$, and $\mathbf{e}_i$ is the i^{th} column of I_{m-k}. True or False? The set $\left\{\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_{m-k}\right\}$ is a basis for $N(A)$.

(h) If B is obtained from A by a single elementary row operation, and $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$ are linearly independent, then $\{\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_k}\}$ are linearly independent.

(i) The number of linearly independent columns of a matrix A is the same as the number of linearly independent rows of A (rank A).

(j) If B is nonsingular, then $N(BA) = N(A)$ (in particular $N(A)$ and the null space of the reduced row achelon of A are identical).

15. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

16. If $A = A^T$, then the eigenvalues λ_i are all real, and $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$.

17. If $A = A^T$, then $\operatorname{tr}(A) = \lambda_1 + \ldots + \lambda_n$.

18. If A^{-1} exits, then $A\mathbf{x} = \lambda \mathbf{x}$ yields $\lambda \neq 0$.

19. If A^{-1} exits, $A = A^T$, and $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.

20. If $A = A^T$, and A^{-1} exits, then A^{-1} is symmetric.

21. If A is an $n \times n$ matrix such that $I + A + A^2 + \ldots + A^n + \ldots$ converges, then

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots.$$

22. If $S = -S^T$ (is skew–symmetric), then $I + S$ is nonsingular.

23. If A and B are nonsingular, and $AB = BA$, then $AB^{-1} = B^{-1}A$.

24. If A is a matrix, $A\mathbf{x}_i = \mathbf{y}_i$, $i = 1, \ldots, n$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ is a linearly independent set, then $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a linearly independent set.

25. True or False? If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a linearly independent set, and $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, then XX^T is nonsingular.

26. True or False? If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a linearly independent set, and $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, then X^TX is nonsingular.

27. Let A and B be $n \times m$ matrices. True or False? If there are matrices P and Q such that $AP = B$ and $A = BQ$, then P and Q are nonsingular.

28. If E is an elementary matrix, then rank $(EA) =$ rank A. If B, C are invertible, then rank $(BA) =$ rank $(AC) =$ rank A.

29. True or False? If A, B are $n \times n$ matrices, then rank $(AB) =$ rank (BA).

30. Let A be an $n \times m$ matrix. If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_p, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_m]$, and $\dim R(A) = p$, then $\dim N(A^T) = m - p$.