MATH221

Final Exam 05/17/18

Total Possible 200

Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (20) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}$. For $\mathbf{b} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$ solve the linear system $A\mathbf{x} = \mathbf{b}$.

   $\mathbf{x} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

2. (20) Let A be an $n \times n$ matrix so that $A^T A = I$. True or False? $\det A^2 = 1$.

   Solution. Since $\det A^T = \det A$ one has $\det A^2 = \det A^T A = \det I = 1$.

   Mark one and explain.
   □ True □ False

3. (20) Let A be an $n \times n$ matrix so that for each vector $\mathbf{v}$ one has $A\mathbf{v} = \mathbf{v}$. True or False? $A = I$.

   Solution. Let $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$. Since $\mathbf{a}_i = A\mathbf{e}_i = \mathbf{e}_i$ one has $A = I$.

   Mark one and explain.
   □ True □ False

4. (60) Let $\mathbf{v}$, $\mathbf{w} \in \mathbf{R}^n$, and $a = \mathbf{v}^T\mathbf{w}$. Consider the $n \times n$ matrix $A = \mathbf{v}\mathbf{w}^T$.

   (a) (20) Show that a and 0 are eigenvalues of A.

   Solution. $A\mathbf{v} = \mathbf{v}\mathbf{w}^T\mathbf{v} = a\mathbf{v}$. If $\mathbf{x} \in \left\{\left(\mathbf{w}^T\right)^{\perp}\right\}$, then $A\mathbf{x} = 0 = 0\mathbf{x}$.

(b) (20) Find an eigenvector $\mathbf{x}$ that corresponds to the eigenvalue a.

Solution. $\mathbf{x} = \mathbf{v}$ (see above).

$\mathbf{x} =$

(c) (20) Find dimension $\dim V_a$ of the eigenspace that corresponds to the eigenvalue a, and dimension $\dim V_0$ of the eigenspace that corresponds to the eigenvalue 0.

Solution. $\dim V_0 = n - 1$, $\dim V_a = 1$ (see above).

$\dim V_a =$
$\dim V_0 =$

5. (80) Let $\mathbf{v},\ \mathbf{w} \in \mathbf{R}^n$ be nonzero vectors, and $A = I + \mathbf{v}\mathbf{w}^T$.

(a) (20) True or False? $A\mathbf{x} = \mathbf{x}$ iff $\mathbf{w}^T\mathbf{x} = 0$.

Solution. $\mathbf{x} = A\mathbf{x} = (I + \mathbf{v}\mathbf{w}^T)\mathbf{x} = \mathbf{x} + \mathbf{v}\mathbf{w}^T\mathbf{x}$.

Mark one and explain.
▫ True ▫ False

(b) (20) Show that 1 is an eigenvalue of A.

Solution. If $\mathbf{x} \in \left\{\left(\mathbf{w}^T\right)^{\perp}\right\}$, then $\mathbf{x} = A\mathbf{x}$.

(c) (20) Show that $1 + \mathbf{w}^T\mathbf{v}$ is an eigenvalue of A.

Solution. $A\mathbf{v} = (I + \mathbf{v}\mathbf{w}^T)\mathbf{v} = (1 + \mathbf{w}^T\mathbf{v})\mathbf{v}$.

(d) (20) Show that A is invertible iff $\mathbf{w}^T\mathbf{v} \neq -1$. In this case find the value a so that $A^{-1} = I + a\mathbf{v}\mathbf{w}^T$.

Solution. $(I + \mathbf{v}\mathbf{w}^T)(I + a\mathbf{v}\mathbf{w}^T) = I + a\mathbf{v}\mathbf{w}^T + \mathbf{v}\mathbf{w}^T + a(\mathbf{w}^T\mathbf{v})\mathbf{v}\mathbf{w}^T$, and $a = -\dfrac{1}{1 + \mathbf{w}^T\mathbf{v}}$.

$a =$

6. (20) Let Let A be an $n \times n$ matrix and $\det A = 0$. True or False? 0 is an eigenvalue of A.

Solution. If $\det A = 0$, then columns of A are linearly dependent, and there is $\mathbf{x} \neq 0$ so that $A\mathbf{x} = 0\mathbf{x}$.

Mark one and explain.
▫ True ▫ False