MATH 221, Spring 2018 - Homework 2 Solutions

Due Tuesday, February 13

Note - Feedback was not provided on the following problems:
Problem 29 from Section 1.3
Problems 24 and 26 from Section 1.4

General Comments:

- Please remember to answer the question that is being asked. For example, if a question asks "Is $\mathbf{v}$ in the span of the columns of A?", your final answer should include either: "yes, $\mathbf{v}$ is in the span of the columns of A" or "no, $\mathbf{v}$ is not in the span of the columns of A". **Simply row-reducing the augmented system $[A\,\mathbf{v}]$ without answering the question is insufficient. Additionally, only stating "yes" or "no" without a justification is insufficient. Always explain your reasoning!**
  - Example: Because there is a pivot in each column of the augmented matrix $[A\,\mathbf{v}]$, there is no solution to the equaion $A\mathbf{x} = \mathbf{v}$. Therefore, $\mathbf{v}$ is not in the span of the columns of A.
- Answering the question of whether or not a vector (for example $\mathbf{b}$) is in the span of the columns of a matrix (for example A) is equivalent to determining if $A\mathbf{x} = \mathbf{b}$ has a solution. **The solution can be unique (no free variables) or there may exist infinitely many solutions (at least one free variable).**
- Note the difference in properties relating to only a matrix A vs. an augmented matrix $[A\,\mathbf{b}]$ (paritcularly the warning after Theorem 4 on page 37 of the text).
- A true/false question always needs supporting evidence. This can either be a reference to a page in the text, a theorem, a definition, or a counterexample.

Section 1.3

Page 32, Problem 12:

Asking whether the vectors form a linear combination of vector $\mathbf{b}$ is equiavelnt to determining if the linear system that forms the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution. The matrix is $\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$ and row operations result in: $-R_1 + R_3 \to R_3$: $\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & 2 \end{bmatrix}$. Because there is a pivot in every row and none in the right-most column, the linear system is consistent, and hence the vectors **do** form a linear combination of the vector $\mathbf{b}$.

Page 32, Problem 14:

Asking whether the vectors formed by the columns of matrix A form a linear combination of vector $\mathbf{b}$ is equiavelnt to determining if the linear system that forms the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution. Therefore, the matrix is $\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$, and performing row operations $2R_1 + R_2 \to R_2$: $\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix}$ $-2R_2 + R_3 \to R_3$:

$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is a consistent system (with infinitely many solutions). Therefore, the vectors of the columns of the matrix A **do** form a linear combination of the vector **b.**

Page 32, Problem 16:

This question is asking for what value of h is $\mathbf{y}$ in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$. In $\mathbb{R}^3$, the span of two nonzero vectors (with neither the multiple of the other) is a plane that contains the two vectors and the origin in addition to all the vectors that can be written as a linear combination of the two vectors. Therefore, to determine when $\mathbf{y}$ is in this plane, determine when the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{y}$ is consistent. To do so, write the system as a linear combination and reduce:

$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix}$ $2R_1 + R_3 \to R_3$: $\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h + -5 \end{bmatrix}$ $-3R_2 + R_3 \to R_3$: $\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h + 4 \end{bmatrix}$. The system will only be consistent when there is no pivot in the right-most column, so $2h + 4 = 0$ in order for there to be no pivot in that position. So, $h = -2$.

Page 32, Problem 23c:

True or False: An example of a linear combination of the vectors $\mathbf{v}_1$and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$. **TRUE**

Consider the linear combination $\frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2$ (on page 28 of the text).

Page 32, Problem 23d:

True or False: The solution set of the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$. **TRUE**

This is defined in the box on page 29 of the text.

Page 32, Problem 23e:

True or False: The set $Span\{\mathbf{u}, \mathbf{v}\}$is always visualized as a plane through the origin. **FALSE**

This is true only when $\mathbf{u}$ and $\mathbf{v}$ are both nonzero with $\mathbf{v}$ not a multiple of $\mathbf{u}$ (as explained on page 30 in the text).

Page 32, Problem 29:

Direct calculation: $\bar{\mathbf{v}} = \frac{1}{4+2+3+5}\left(4\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 17/14 \\ -34/14 \\ 16/14 \end{bmatrix} = \begin{bmatrix} 17/14 \\ -17/7 \\ 8/7 \end{bmatrix}$

Section 1.4

Page 40, Problem 2:

The product is **not defined** because the order of the matrix is 3x1 and the order of the vector is 2x1. The number of columns of the matrix (1) does not equal the number of entries of the vector (2).

Page 40, Problem 4:

The product **is defined** because the order of the matrix is 2x3 and the vector is 3x1 (so the number of columns (3) in the matrix is equal to the number of entries in the vector). The order of the product should be 2x1.

a. Using the definition, as in Example 1 on page 35:

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

b. Using the row-vector rule (explained on page 38):

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 3(2) + -4(1) \\ 3(1) + 2(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Page 40, Problem 6:

This exercise is similar to part (a) of problem 4. Use the elements of the vector as scalars for the columns of the matrix:

$$-3 \cdot \begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5 \cdot \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

Page 40, Problem 8:

This is similar to the previous exercise, but now write the column vectors as a 2x4 matrix, the scalars as a 4x1 column-vector, and keep the left-side of the equation as a two-column vector:

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Page 40 Problem 9:

Vector Equation: $x_1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ Matrix Equation: $\begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

Page 40, Problem 12:

Augmented Matrix: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix}$ Row-Reduction: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ The solution, as a vector: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$

Page 40, Problem 13:

To answer this question, determine if $\mathbf{u}$ is in the Span of these columns, determine if $\mathbf{u}$ is a linear combination of the columns of A. That is, determine if A$\mathbf{x} = \mathbf{u}$ has a solution. The augmented matrix is $\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

and row-reduction yields: $\begin{bmatrix} 1 & 1 & 4 \\ 3 & -5 & 0 \\ -2 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 0 & -8 & -12 \\ 0 & 8 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there is no pivot in the last column, a solution exists, so **u is in the plane in $\mathbb{R}^3$ spanned by the columns of A**.

Page 40, Problem 14:

This question is answered in the same way as above. That is, determine if A$\mathbf{x} = \mathbf{u}$ has a solution.

The augmented matrix is $\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix}$ and row-reduction yields:

$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. Because there is a pivot in the last column, no solution exists, so **u is NOT in the subset of $\mathbb{R}^3$ spanned by the columns of A**.

Page 41, Problem 22:

The matrix formed by these vectors is $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} -3 & 9 & -6 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

It is clear that there is a pivot in each row, so the vectors span $\mathbb{R}^3$ by Theorem 4 of this section.

Page 42, Problem 34:

We know $\mathbf{v}_1 = A\mathbf{u}_1$ and $\mathbf{v}_2 = A\mathbf{u}_2$ are consistent and $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. So, $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 = A\mathbf{u}_1 + A\mathbf{u}_2$. By Theorem 5a of this section, $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2 = A(\mathbf{u}_1 + \mathbf{u}_2)$. Therefore, $\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2$ is a solution to $A\mathbf{x} = \mathbf{w}$.

Page 41, Problem 35:

Assume A$\mathbf{y} = \mathbf{z}$ is true. Then, $5\mathbf{z} = 5\text{A}\mathbf{y} = \text{A}(5\mathbf{y})$ (by Theorem 5b on page 39). Let $\mathbf{x} = 5\mathbf{y}$. Then, A$\mathbf{x} = 5\mathbf{z}$ is also consistent.

Section 1.5

Page 47, Problem 2:

Use row operations on the augmented matrix: $\begin{bmatrix} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & -7 & 5 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$. Because there is a pivot in every column of the coefficient matrix, there are no free variables, **so the system has only the trivial solution**.

Page 47, Problem 8:

In order to solve this problem, put the matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0} \end{bmatrix}$ (where $\mathbf{a}_1$, etc. are the columns of A) in reduced echelon form: $\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$, which is equivalent to the system $\begin{array}{c} x_1 - 2x_3 - 7x_4 = 0 \\ x_2 + 2x_3 - 4x_4 = 0 \end{array}$. It is clear that the basic variables are x_1 and x_2 while the free varaibles are x_3 and x_4. Solving for the free variables results in: $\begin{array}{c} x_1 = 2x_3 + 7x_4 \\ x_2 = -2x_3 + 4x_4 \end{array}$. Writing in parametric vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Page 47, Problem 10:

This is the same process as problem 8 in this section: $\begin{bmatrix} -1 & -4 & 0 & -4 & 0 \\ 2 & -8 & 0 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{array}{c} x_1 = -4x_4 \\ x_2 = 0 \end{array}$. The basic variables are x_1 and x_2 while the free variables are x_3 and x_4. The parametric vector form is: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Page 47, Problem 12:

This is the same process as the previous two problems: $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{array}{c} x_1 = 2x_2 - 3x_3 - 29x_5 \\ x_4 = -4x_5 \\ x_6 = 0 \end{array}$. The basic variables are x_1, x_4, and x_6 .

The free variables are x_2, x_3, and x_5. The solution in parametric vector form is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Page 47, Problem 13:

As vectors, this line is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$, which is a line through $\begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$.

Page 47, Problem 15:

First, realize that the second equation is the first equation shifted by 2. Solving the first equation for x_1 results in

$x_1 = -5x_2 + 3x_3$. In vector form, this is the same as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, which is a plane

through the origin spanned by $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. The solution to the second equation is:

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, which is a parallel plane through $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ instead of $\mathbf{0}$.

Page 47, Problem 18:

The system as an augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix}$ and row reduction yields: $\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the parametric solution being $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$.

This solution is a line through $\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution to the homogenous equation in Exercise 6.

Page 48, Problem 35:

By inspection, the second column of A, $\mathbf{a}_2 = 3\mathbf{a}_1$. Therefore, one **nontrivial** (not $\mathbf{0}$) solution is

$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ or $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Page 48, Problem 38:

By Theorem 5b on page 39, $A(c\mathbf{w}) = cA\mathbf{w}$. Since $\mathbf{w}$ satisfies A$\mathbf{x} = \mathbf{0}$, A$\mathbf{w} = \mathbf{0}$. So, $cA\mathbf{w} = c\mathbf{0} = \mathbf{0}$, so $A(c\mathbf{w}) = \mathbf{0}$.

Section 2.1

Page 100, Problem 3:

To begin, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -3 & 5 \end{bmatrix}$ and

$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -15 \\ 9 & -6 \end{bmatrix}$

Page 100, Problem 5:

a. $A\mathbf{b}_1 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 26 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ -19 \end{bmatrix}$ So, $AB = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1(4)+3(-2) & -1(-2)+3(3) \\ 2(4)+4(-2) & 2(-2)+4(3) \\ 5(4)+-3(-2) & 5(-2)+-3(3) \end{bmatrix} = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$

Page 100, Problem 6:

a. $A\mathbf{b}_1 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 12 \\ 3 \end{bmatrix}$ $A\mathbf{b}_2 = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 22 \\ -22 \\ -2 \end{bmatrix}$ So, $AB = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

b. $AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4(1)+-3(3) & 4(4)+-3(-2) \\ -3(1)+5(3) & -3(4)+5(-2) \\ 0(1)+1(3) & 0(4)+1(-2) \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

Page 100, Problem 12:

Because A is 2x2 and B is 2x2, our new matrix of all zeros will also be 2x2. Essentially, we want to solve

$\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ with non-zero columns. Multiplying these matrices results in a linear system:

$\begin{matrix} 3a-6c=0 \\ 3b-6d=0 \\ -2a+4c=0 \\ -2b+4d=0 \end{matrix}$, which can be broken into two separate systems: $\begin{matrix} 3a-6c=0 \\ -2a+4c=0 \end{matrix}$ and $\begin{matrix} 3b-6d=0 \\ -2b+4d=0 \end{matrix}$.

Using row reduction, $\begin{bmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $a = 2c$ and $b = 2d$. Answers will vary.

An example is $c = 1$, $d = 1$ so $a = b = 2$: $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Page 101, Problem 24:

Remember, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $D = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix}$. By definition of matrix multiplication, the columns of AD are equivalent to $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$, respectively. In order for $AD = I_3$, the systems generated by $A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$ must each have at least one solution. Since the columns of A span $\mathbb{R}^3$, each of theses systems do have at least one solution (see Theorem 4 in Section 1.4). So, the matrix D is found by selecting one of the solutions from each of the systems ($A\mathbf{d}_1$, $A\mathbf{d}_2$, and $A\mathbf{d}_3$) and using it as the columns of D.

Page 101, Problem 26:

Let $\mathbf{b} \in \mathbb{R}^m$ be arbitrary ($\mathbf{b}$ is an m x 1 matrix or vector). Assume $AD = I_m$ is true. Then, multiplying by $\mathbf{b}$ yields $AD\mathbf{b} = I_m\mathbf{b}$, which implies $AD\mathbf{b} = \mathbf{b}$ (in matrix algebra I_m is treated like the number 1). Because the order of the matrices is defined, $A(D\mathbf{b}) = \mathbf{b}$ (by Theorem 2 of this section on page 97). The product $D\mathbf{b}$ is a vector which can be written as $\mathbf{x} = D\mathbf{b}$. So, $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$. By Theorem 4 in Section 1.4, since $A\mathbf{x} = \mathbf{b}$ is true for every $\mathbf{b}$ in $\mathbb{R}^{\mathrm{m}}$, A has a pivot position in every row. Because each pivot is in a different column, A must have at least as many columns as rows.