MATH 221, Spring 2018 - Homework 3 Solutions

Due Tuesday, February 20

General Comments:

- Properties of matrix algebra are different than the properties of real-number algebra. You should read pages 93-98 of the text for a more detailed explanation, but below are some important differences.
- Whereas in real-number algebra for any numbers x and y it is true that $xy = yx$, in matrix algebra it is **not always true** that for any two matrices A and B that $AB = BA$. For example, if we have the expression ABC involving matrices, **IT IS NOT TRUE THAT** $ABC = CAB$ or $ABC = ACB$.
- There is no "division" operation in matrix algebra. For example, given a matrix A, there is no matrix $\frac{1}{A}$. The inverse of a matrix (if it exists), is denoted A^{-1}, NOT $\frac{1}{A}$.
- You cannot "cancel" matrices out of expressions. For example, if we have $ABC = DCP$, **it is not true (in general) that** $AB = DP$**.**
- In real-number algebra for any numbers x and y if $xy = 0$, then either $x = 0$ or $y = 0$, or $x = y = 0$. In matrix algebra it is **not always true** that for any two matrices A and B if $AB = 0$, then either $A = 0$, or $B = 0$, or $A = B = 0$.
  - Consider the following example. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Obviously, neither A nor B is the zero matrix, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- If we have an expression $AB = BC$, and we wish to multiply both sides of the equation by a matrix D, we must multiply D on the same side of each equation. If we right-multiply, we get $ABD = BCD$. If we left-multiply, we get $DAB = DBC$. You cannot do both. For example, **it is not true, in general, that** $ABD = DBC$.

Section 1.7

Page 60, Problem 6:

Determine if A$\mathbf{x} = \mathbf{0}$ has only the trivial solution:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ -4 & -3 & 0 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & -20 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because there are no free variables, the system has only the trivial solution, so **the columns of A form a linearly independent set.**

Page 60, Problem 8:

You could use the same process as above, but notice that there are 4 vectors in $\mathbb{R}^3$ (because the matrix is 3 x 4).

By Theorem 8 of this section, **the vectors are linearly dependent** (there must be at least one free variable, if a solution exists).

Page 61, Problem 14:

In order for the vectors to be linearly dependent, the system A$\mathbf{x} = \mathbf{0}$ (where A is a matrix formed by the column vectors) must have a nontrivial solution.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 1 & 0 \\ -4 & 6 & h & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -6 & 8+h & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 38+h & 0 \end{bmatrix}.$$

A nontrivial solution exists when there is a free variable. Therefore, a nontrivial solution exists for $h = -38$.

Page 61, Problem 18:

This a set of 4 vectors in $\mathbb{R}^2$. By Theorem 8, because p = 4 > 2 = n, the **set of vectors is linearly dependent.**

Page 61, Problem 20:

By Theorem 9, any set that contains the zero vector is linearly dependent. Thus, **this set is linearly dependent.**

Page 61, Problem 21a:

True or False: The columns of a matrix A are linearly independent if the equation A$\mathbf{x} = \mathbf{0}$ has the trivial solution.

FALSE - A homogenous system always has the trvial solution (as explained on page 56). The question of linear independence is whether the trivial solution is the **only** solution.

Page 61, Problem 21b:

True or False: If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

FALSE - Not all vectors need to be linear combinations of each other. At least one of the vectors needs to be a linear combination of the others (see Thereom 7 and the following warning on page 58).

Page 61, Problem 21c:

True or False: The columns of any 4 x 5 matrix are linearly dependent.

TRUE - In this case, there are 5 vectors in $\mathbb{R}^4$. By Theorem 8 in this section, because n = 4 < 5 = p, the set of vectors formed by the columns of this matrix are linearly dependent.

Page 61, Problem 21d:

If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in $Span\,\{\mathbf{x}, \mathbf{y}\}$.

TRUE - Because $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent but $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, $\mathbf{z}$ must be a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Thus, $\mathbf{z}$ must be in $Span\,\{\mathbf{x}, \mathbf{y}\}$.

Page 61, Problem 30:

a) Complete the blank: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has **n** pivot columns.

b) The columns of A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$ is the only solution, which is true if and only if there are no free variables, which happens if and only if every column of A has a pivot.

Section 2.2

Page 109, Problem 4:

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \; A^{-1} = \frac{1}{-12+16} \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

Page 109, Problem 9a:

True or False: In order for a matrix B to be the inverse of A, the equations AB = I and BA = I must both be true.

TRUE - This is the definition of invertible on page 103.

Page 109, Problem 9b:

True or False: If A and B are n x n and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

FALSE - By Theorem 6 on page 105, $(AB)^{-1} = B^{-1}A^{-1}$, which does not always equal $A^{-1}B^{-1}$.

Page 109, Problem 9c:

True or False: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

FALSE - By Theorem 4 of this section, a 2 x 2 matrix is invertible if and only if $ad - bc \neq 0$.

The expression $ab - cd$ reveals nothing about the invertibility of a matrix.

For example, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow ab - cd = 1 - 0 \neq 0$, but the matrix is not invertible because $ad - bc = 0$.

Page 109, Problem 9d:

True or False: If A is an invertible n x n matrix, then the equation A$\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^{\mathrm{n}}$.

TRUE - This follows from Theorem 5 of this section on page 104.

Page 110, Problem 14:

Because $(B - C)$ is an m x n matrix, D must be an n x n matrix (because the product $(B - C)D$ is defined

and D is invertible). Thus, 0 is an m x n matrix. Beacuse D is invertible,

$(B - C)DD^{-1} = 0 \cdot D^{-1} \Rightarrow (B - C)I_n = 0$, where 0 is still an m x n matrix because D^{-1} is still n x n.

Thus, $B - C = 0$ because I_n is essentially 1. Thus, $B - C + C = 0 + C \Rightarrow B + (-C + C) = 0 + C \Rightarrow B = C$.

Page 110, Problem 16:

Because A and B are both n x n matrices, their products and inverses (if they exist) are also n x n.

Using the hint, let $C = AB$ and solve for A: $CB^{-1} = ABB^{-1} \Rightarrow CB^{-1} = A$, but $C = AB$.

Therefore, A is the product of invertible matrices. By Theorem 6 of this section, A must also be invertible.

Page 110, Problem 18:

Because the order of all matrices is n x n, their products and inverses (if they exist) are also n x n.

Because B is invertible, $ABB^{-1} = BCB^{-1} \Rightarrow AI_n = BCB^{-1} \Rightarrow A = BCB^{-1}$.

Page 110, Problem 31:

To find the inverse, use the algorithm on page 108:

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

So, the inverse is $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.