# MATH 221, Spring 2018 - Homework 4 Solutions

Due Tuesday, Februrary 27

## Section 1.8

Page 68, Problem 2:

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{a}{3} \\ \frac{b}{3} \\ \frac{c}{3} \end{bmatrix}$$

Page 68, Problem 4:

We look for $\mathbf{x}$ such that $\mathbf{b} = T(\mathbf{x}) = A\mathbf{x}$, which is equivalent to solving the system represented by the augmented matrix:

$\begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix}$. Row reduction yields $\begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Therefore, the solution is unique and $\mathbf{x} = \begin{bmatrix} -17 \\ -7 \\ -1 \end{bmatrix}$.

Page 68, Problem 9:

We look for $\mathbf{x}$ such that $\mathbf{0} = T(\mathbf{x}) = A\mathbf{x}$, which is equivalent to solving the system represented by the augmented matrix:

$\begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 2 & -4 & 4 & -4 & 0 \end{bmatrix}$. Row reduction yields $\begin{bmatrix} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Since the system is overdetermined, there is a free variable ($x_3$) and the solution is given by: $\mathbf{x} = t \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, where $t$ is an arbitrary scalar in $\mathbb{R}$.

Page 68, Problem 17:

Linear transformations preserve operations of vector addition and scalar multiplication (page 66). Therefore,

$$T(2\mathbf{u}) = 2T(\mathbf{u}) = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$T(3\mathbf{v}) = 3T(\mathbf{v}) = 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$T(2\mathbf{u} + 3\mathbf{v}) = T(2\mathbf{u}) + T(3\mathbf{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Page 69, Problem 23:

**a.** When $b = 0$, then $f(x) = mx$. So, for any $x,\ y \in \mathbb{R}$ and scalars $a$ and $b$, we have:

$f(ax + by) = m(ax + by) = m(ax) + m(by) = a(mx) + b(my) = af(x) + bf(y)$ by properties of Real Numbers.

**b.** When $b \neq 0$, $f(0) = m(0) + b = b \neq 0$, which is a violation of the property that linear transformations always map zero to zero.

**c.** $f$ is called a linear function because its graph is a straight line (demonstrating a linear relationship)

Page 69, Problem 26:

**a.** Refering to the figure on page 47, because **q - p** is parallel to line M, and **p** lies on M, a parametric equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$. Expanding this expression yields $\mathbf{x} = \mathbf{p} + t\mathbf{q} - t\mathbf{p} \Rightarrow \mathbf{x} = \mathbf{p} - t\mathbf{p} + t\mathbf{q} \Rightarrow \mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$.

**b.** Because $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$, $T(\mathbf{x}) = T((1-t)\mathbf{p} + t\mathbf{q})$, then by definition of linear transformations,

$T((1-t)\mathbf{p} + t\mathbf{q}) = T((1-t)\mathbf{p}) + T(t\mathbf{q}) = (1-t)T(\mathbf{p}) + tT(\mathbf{q})$

If **p** and **q** are distinct, then this equation is representative of the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$ (like the equation found in part a). Otherwise, $T(\mathbf{p}) = (1-t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p}) - tT(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$, which is a single point. (the same is true for $T(\mathbf{q})$)

Page 69, Problem 27:

$T(\mathbf{x}) = T(s\mathbf{u} + t\mathbf{v}) = sT(\mathbf{u}) + tT(\mathbf{v})$ such that $s,\ t \in \mathbb{R}$

The set of images is *Span* $\{T(\mathbf{u}),\ T(\mathbf{v})\}$. If $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is linearly independent, then *Span* $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is linearly dependent (one is a multiple of the other and not both zero), then *Span* $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then *Span* $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

Page 69, Problem 30:

Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ span $\mathbb{R}^{\mathrm{n}}$, then any $\mathbf{x} \in \mathbb{R}^{\mathrm{n}}$ can be written as $\mathbf{x} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n$, for constants $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Then, $T(\mathbf{x}) = T(\alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n) = \alpha_1 T(\mathbf{v}_1) + \ldots + \alpha_1 T(\mathbf{v}_n) = \alpha_1\mathbf{0} + \ldots + \alpha_n\mathbf{0} = \mathbf{0}$.

Page 69, Problem 32:

If $T$ were linear then $T(c\mathbf{x}) = cT(\mathbf{x})$. Use any counterexample to show this is not true.

$T((0,\ 1)) = (-2,\ -4)$, but $T(-1 \cdot (0,\ 1)) = T((0,\ -1)) = (-2,\ 4) \neq -1 \cdot T((0,\ 1)) = (2,\ 4)$

Page 70, Problem 36:

Begin with the hint. We know that because $\{T(\mathbf{u}),\ T(\mathbf{v})\}$ is linearly dependent, there exist scalars $c_1$ and $c_2$ (not both zero), such that $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$. Because $T$ is linear, this becomes $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. Let $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$.

Because $c_1$ and $c_2$ are **not both** zero (one may be 0) and $\{\mathbf{u},\ \mathbf{v}\}$ is linearly independent (which implies neither **u** or $\mathbf{v}$ are $\mathbf{0}$), $c_1\mathbf{u}+c_2\mathbf{v} \neq \mathbf{0}$. Thus, $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

## Section 1.9

Page 78, Problem 11:

The transformation maps $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_1$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_2$, which in matrix form is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

This is the same as a rotation through $\pi$ radians because $\begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Since a linear transformation is completely determined by what it does to the columns of the identity matrix (Theorem 10 of this section), the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

Page 78, Problem 15:

The matrix entries are the coefficients of the variables on the right-hand side of the equation: $\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Page 78, Problem 22:

In this problem, we will use the fact that $T(\mathbf{x}) = A\mathbf{x}$. Because $T : \mathbb{R}^2 \to \mathbb{R}^3$, we know $\mathbf{x}$ is a 2 x 1 vector and the matrix A must be 3 x 2. Therefore the set up of the transformation should be of the form:

$T(\mathbf{x}) = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix}$. The missing entries of the matrix A are the coefficients of the variables on the right-hand side of the equation. Therefore: $T(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Because we are looking for $\mathbf{x}$such that

$T(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, we solve the system: $\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Page 79, Problem 26:

In order to answer this question, note that the transformation matrix is $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$.

Because there are more columns than rows, the columns must be linearly dependent. Therefore, $T$ **is not one-to-one**.

If we row-reduce the matrix, $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$, we see that there is a pivot in every row.

Thus, the columns of A span $\mathbb{R}^2$. Hence, $T$ **is onto.**

Page 79, Problem 34:

Using the hint, let $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^p$ be arbitrary vectors and let $c,\ d \in \mathbb{R}$ be arbitrary scalars.

Because $S$ is linear, $T(S(c\mathbf{u} + d\mathbf{v})) = T(cS(\mathbf{u}) + dS(\mathbf{v}))$. Because $T$ is linear, $T(cS(\mathbf{u}) + dS(\mathbf{v})) = cT(S(\mathbf{u})) + dT(S(\mathbf{v}))$.

Therefore, $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation.