MATH 221, Spring 2018 - Homework 5 Solutions

Due Tuesday, March 13

Section 2.3

Page 115, Problem 2:

$$A = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$$

Notice that $\mathbf{a}_2 = -\frac{1}{2}\mathbf{a}_1$ where $\mathbf{a}_i$ is the column vector of the matrix A. Thus, the columns are linearly dependent. By Theorem 8 of this section, the **matrix is singular (nonivertible).** Also, notice that the determinant is equal to 0. So, by Theorem 4 of the previous section, the matrix is singular.

Page 115, Problem 4:

$$A = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix} \quad A^T = \begin{bmatrix} -5 & 0 & 1 \\ 1 & 0 & 4 \\ 4 & 0 & 9 \end{bmatrix}$$

Notice that the columns of A^T are linearly dependent because the zero vector is a member of the set.

Thus, A^T is singular (noninvertible). Hence A is singular (nonivertible), by Theorem 8.

Also, because A contains a row of zeros, it cannot be reduced to the identity matrix.

Therefore, by Theorem 8, it is signular (noninvertible).

Page 115, Problem 8:

$$A = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the matrix is in echelon form, it is clear that there is a pivot in every row.

Hence, the matrix is invertible by Theorem 8.

Page 115, Problem 11a:

True or False: If the equation A$\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

TRUE: Because (d) of Theorem 8 is true, (b) must also be true.

Page 115, Problem 11d:

True or False: If the equation $\mathrm{A}\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

TRUE: Because (d) of Theorem 8 is false, (c) must also be false. An $n \times n$ matrix can never have more than n pivot positions, so it must have fewer than n.

Page 115, Problem 11e:

True or False: If A^T is not invertible, then A is not invertible.

TRUE: Because (l) of Theorem 8 is false, (a) must also be false.

Page 115, Problem 12a:

True or False: If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.

TRUE: Because (k) of Theorem 8 is true, (j) is also true. Because $AD = I$, $D = A^{-1}$, so $DA = A^{-1}A = I$.

Page 115, Problem 12b:

True or False: If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^{\mathrm{n}}$ into $\mathbb{R}^{\mathrm{n}}$, then the row reduced echelon form of A is I.

FALSE: In order for this to follow from Theorem 8, $\mathbf{x} \mapsto A\mathbf{x}$ must map $\mathbb{R}^{\mathrm{n}}$ **onto** $\mathbb{R}^{\mathrm{n}}$, not into.

Page 115, Problem 12c:

True or False: If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^{\mathrm{n}}$.

TRUE: Because (e) of Theorem 8 is true, (h) must also be true.

Page 115, Problem 21:

Notice that on page 112, in the paragraph at the end of the page, it says (g) in Theorem 8 could be rewritten as

"The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^{\mathrm{n}}$."

In problem 21, this statement is false, thus (h) of Theorem 8 must also be false, so the columns of C **do not span** $\mathbb{R}^{\mathrm{n}}$.

Page 115, Problem 27:

Assume AB is invertible. Then, by Theorem 8(k) of this section, there exists an $n \times n$ matrix W such that $ABW = I$.

By properties of matrices (and because the order is defined), $ABW = A(BW) = I$.

Because A is square, let $BW = D$. Thus, by Theorem 8(k), A is invertible.

Page 115, Problem 30:

Since statement (f) of the IMT is false, we know all other parts of the theorem are false. Thus, the transformation is not onto, A is not invertible, and the transformation is not invertible (by Theorem 9).

Page 115, Problem 39:

Because T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the standard matrix A is invertible, by Theorem 8 of this section.

Hence, by Theorem 9 of this section, T is invertible and A^{-1} is the standard matrix of T^{-1}.

Thus, by Theorem 8 of this section, the columns of A^{-1} are linearly independent and span $\mathbb{R}^n$.

By Theorem 12 in Section 1.9, this shows that T^{-1} is a one-to-one mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Section 4.1

Page 196, Problem 16:

It is clear that W is not a vector space because it can never contain the zero vector (the first entry is always 1).

Page 196, Problem 21:

The set H is a subspace of M_{2x2} because:

1) If $a = b = d = 0$, the zero vector is contained in the space.

Let $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ be two arbitrary matrices in H.

2) Then, $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, so H is closed under addition.

3) Let β be an arbitrary scalar. Then, $\beta \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} \beta a_1 & \beta b_1 \\ 0 & \beta d_1 \end{bmatrix}$, which is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

So H is closed under scalar multiplication.

Page 196, Problem 22:

The set M_{2x4} is the set of all matrices of the form $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ where the entries are arbitrary.

This set is a subspace (as stated in the problem).

Let the matrix F be $F = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix}$ where the entries are fixed.

The set $H = \{A \in M_{2x4} \,:\, FA = 0\}$ is a subset of M_{2x4}. To show H is a subspace:

1) Because $F0 = 0$, $0 \in H$.

2) Let A_1 and A_2 be arbitrary matrices in H. Then, $F(A_1) = 0$ and $F(A_2) = 0$.

Because$F(A_1 + A_2) = F(A_1) + F(A_2) = 0 + 0 = 0$. Thus, $A_1 + A_2 \in H$, so H is closed under addition.

3) Let $A \in H$ and $c \in \mathbb{R}$ be arbitrary. Thus, $FA = 0$. So, $F(cA) = cFA = c(FA) = 0$.

Thus, $cA \in H$, so H is closed under scalar multiplication.

Page 197, Problem 32:

To show $H \cap K$ is a subspace, check the three conditions:

1) Because H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Thus, $\mathbf{0} \in H \cap K$.

2) Let $\mathbf{u} \in H \cap K$ and $\mathbf{v} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$ and $\mathbf{v} \in H$ and $\mathbf{v} \in K$.

Because H and K are subspaces, $\mathbf{u} + \mathbf{v} \in H$ and $\mathbf{u} + \mathbf{v} \in K$. Thus, $\mathbf{u} + \mathbf{v} \in H \cap K$.

3) Let $c \in \mathbb{R}$ and $\mathbf{u} \in H \cap K$ be arbitrary. Then, $\mathbf{u} \in H$ and $\mathbf{u} \in K$.

Because H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{u} \in K$. Thus, $c\mathbf{u} \in H \cap K$.

An example in $\mathbb{R}^2$ to show $H \cup K$ is not always a subspace would be $H = \{(x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, y) : y \in \mathbb{R}\}$

(the x-axis and y-axis, respectively). Let $\mathbf{u}$ =$(1, 0) \in H \cup K$ and $\mathbf{v} = (0, 1) \in H \cup K$.

Then, $\mathbf{u} + \mathbf{v} = (1, 1)$, which is not in H or in K, so it is not in $H \cup K$.

Thus, $H \cup K$ is not closed under addition and is therefore not a subspace.

Page 197, Problem 30:

Assume $c\mathbf{u} = \mathbf{0}$ for some non-zero scalar c. Since c is non-zero, we know there exists $c^{-1} = \frac{1}{c}$ such that $c^{-1}c = 1$.

Thus, $\mathbf{u} = (c^{-1}c)\mathbf{u} = c^{-1}(c\mathbf{u}) = c^{-1}\mathbf{0}$ because we assume $c\mathbf{u} = \mathbf{0}$. So, we get $\mathbf{u} = c^{-1}\mathbf{0} = \mathbf{0}$ by Property 2.