# MATH 221, Spring 2018 - Homework 7 Solutions

Due Tuesday, April 3

## Section 4.3

Page 213, Problem 3:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$,which reduces to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there are only two pivot positions, **the set of vectors are neither linearly independent nor span** $\mathbb{R}^3$, thus **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

Page 213, Problem 8:

The matrix whose columns are the given set of vectors is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix}$. Because there are four columns, **the set cannot be linearly indpendent in** $\mathbb{R}^3$. Thus, **the set of vectors do NOT form a basis of** $\mathbb{R}^3$**.**

To determine if the set of vectors span $\mathbb{R}^3$, row-reduce the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because there is a pivot position in each row, **the set of vectors do span** $\mathbb{R}^3$**.**

Page 213, Problem 13:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1 and 2 (look at matrix $B$, which is in row echelon form). Use these columns from matrix $A$ to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$. To find a basis for Nul$A$, write the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables ($x_3$ and $x_4$): $\mathbf{x} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis for Nul$A$ is $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 214, Problem 14:

To find a basis for Col$A$, use Theorem 6 of this section. Notice that the pivot positions are in columns 1, 3, and 5 (look at matrix$B$, which is in row echelon form). Use these columns from matrix $A$ to form a basis. Therefore, a basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix} \right\}$. To find a basis for Nul$A$, we need the general solution to $A\mathbf{x} = \mathbf{0}$ in terms of the free variables ($x_2$ and $x_4$). Because matrix $B$ is only in row echelon form, reduce it to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \ \mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \text{ Thus a basis for Nul}A \text{ is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Page 214, Problem 21b:

True or False: If $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for $H$.

**FALSE**: The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent.

Page 214, Problem 21c:

True or False: The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^{\text{n}}$.

**TRUE**: Because the matrix is invertible, the columns span $\mathbb{R}^{\text{n}}$ and are linearly independent (by the Invertible Matrix Theorem). Hence, the columns form a basis for $\mathbb{R}^{\text{n}}$.

Page 214, Problem 21d:

True or False: A basis is a spanning set that is as large as possible.

**FALSE**: A basis is a spanning set that is as small possible (read "Two Views of a Basis" on p. 212).

Page 214, Problem 22a:

True or False: A linearly independent set in a subspace $H$ is a basis for $H$.

**FALSE**: In order to be a basis, the set must also span $H$ (by definition).

Page 214, Problem 22b:

True or False: If a finite set $S$ of nonzero vectors spans a vector space $V$, then some subset of $S$ is a basis for $V$.

**TRUE**: By the Spanning Set Theorem, removing linearly dependent vectors in $S$ will still result in a spanning set (this new set is a subset of $S$). Because the new set will eventually only contain linearly independent vectors, the set will be a basis for $V$.

Page 213, Problem 22e:

True or False: If $B$ is an echelon form of a matrix $A$, then the pivot columns of $B$ form a basis for Col$A$.

**FALSE**: The pivot columns in $B$ tell which columns in matrix $A$ form the basis for Col$A$ (see the warning after Theorem 6 on page 212).

Page 214, Problem 25:

While it might seem that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a spanning set for $H$, it is not. Notice that $H$ is a subset of Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, there are vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in $H$, such as $\mathbf{v}_1$ and $\mathbf{v}_3$ (the second and third elements of these vectors are not equal). Therefore, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $H$, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be a basis for $H$.

Page 215, Problem 30:

Since $k > n$, there exist more vectors than there are entries in each vector, so the set is linearly dependent by Theorem 8. Since the set is not linearly independent, it cannot be a basis for $\mathbb{R}^n$.