MATH 221, Spring 2018 - Homework 8 Solutions

Due Tuesday, April 17

Section 4.4

Page 222, Problem 3:

Let $\mathcal{B}$=[$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$]. Then, $\mathbf{x} = 1\mathbf{b}_1 + 0\mathbf{b}_2 + -2\mathbf{b}_3 = 1\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} + -2\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$.

Page 222, Problem 7:

In this problem, we are solving the equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for the coordinates c_1, c_2, and c_3. In this problem, this equation is represented by $\begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, which amounts to solving the augmented system $\begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix}$. Row-reducing yields $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

So, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

Page 223, Problem 10:

As stated in this section (on page 219), the matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ is the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Therefore, $P_{\mathcal{B}} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$.

Page 223, Problem 14:

Any polynomial $a + bt + ct^2$ in $\mathbb{P}_2$ can be written in vector form as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Therefore, the set $\mathcal{B}$ as a set of vectors is $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$ and the vector $\mathbf{p}$ is $\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$. Solve the augmented system $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{bmatrix}$.

The solution in reduced-echelon form is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

Page 223, Problem 22:

Let $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & ... & \mathbf{b}_n \end{bmatrix}$ (which is an $n \times n$ matrix because its columns form a basis for $\mathbb{R}^n$). By definition, $\mathbf{x} = P_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$ which is a transformation of $[\mathbf{x}]_{\mathcal{B}}$ to $\mathbf{x}$. Because the columns of $P_{\mathcal{B}}$ are linearly independent (they form a basis for $\mathbb{R}^n$), $P_{\mathcal{B}}$ is invertible. Thus, left-side multiplication of $P_{\mathcal{B}}^{-1}$ results in $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, which is a transformation of $\mathbf{x}$ to $[\mathbf{x}]_{\mathcal{B}}$ ($\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$). Therefore, take $A = P_{\mathcal{B}}^{-1}$.

Page 222, Problem 26:

Assume $\mathbf{w}$ is a linear combination of $\mathbf{u}_1$, ..., $\mathbf{u}_p$. Then, there exist scalars c_1, ..., c_p so that $\mathbf{w} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p$. Since the coordiante mapping $[\mathbf{w}]_{\mathcal{B}}$ is a linear transformation (Theorem 8), it follows that $[\mathbf{w}]_{\mathcal{B}} = c_1\,[\mathbf{u}_1]_{\mathcal{B}} + ... + c_p\,[\mathbf{u}_p]_{\mathcal{B}}$. So, $[\mathbf{w}]_{\mathcal{B}}$ must be a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}$, ..., $[\mathbf{u}_p]_{\mathcal{B}}$. Since the transformation is one-to-one, the converse must be true.

Section 4.5

Page 229, Problem 3:

Any vector in the subspace can be written as $a\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$. Thus, $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right\}$ spans the subspace. To determine if this set is linearly independent, solve the matrix equation $\begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$.

The matrix reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the only solution is the trivial solution, so the columns are linearly independent. Therefore, a basis for the subspace is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right\}$. Because there are three vectors in the basis, the dimension of the subspace is 3.

Page 229, Problem 8:

The equation can be rewritten as $a = 3b - c$. Thus, any vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in the subspace can be written as $b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Thus, the set $S = \left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ spans the subspace. It is clear that the set is linearly independent, but to verify that, reduce the matrix formed by the column vectors

$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which shows the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, so the columns are linearly independent. Thus, S is a basis with dimension 3.

Page 229, Problem 10:

Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 15 \end{bmatrix}$. It is clear that the set of these vectors is linearly dependent because $\mathbf{v}_2 = -2\mathbf{v}_1$ and $\mathbf{v}_3 = -3\mathbf{v}_1$. By the Spanning Set Theorem, the set $\{\mathbf{v}_1\}$ still spans $\mathbb{R}^2$ and because the set is linearly independent, it is also a basis for $\mathbb{R}^2$, so the dimension is 1.

Page 229, Problem 14:

Because there are three free variables, the dimension of Nul$A$ is 3 and because there are four pivot positions, the dimension of Col$A$ is 4.

Page 229, Problem 15:

Because there are two free variables, the dimension of Nul$A$ is 2 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 17:

Because there are no free variables, the dimension of Nul$A$ is 0 and because there are three pivot positions, the dimension of Col$A$ is 3.

Page 229, Problem 19a:

True or False: The number of pivot columns of a matrix equals the dimension of its column space.

TRUE: This is stated in the box on page 228 before Example 5.

Page 229, Problem 19d:

True or False: If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Page 229, Problem 20d:

True or False: If $\dim V = n$ and if S spans V, then S is a basis for V.

FALSE: The set must have exactly n vectors to be a basis for V.

Section 4.6

Page 236, Problem 2:

Because rank$A = \dim(\text{Col}A)$, and since there are 3 pivot positions, rank$A = 3$. Because A is a 4×5 matrix, $\dim(\text{Nul}A) + \text{rank}A = 5$. Thus, $\dim(\text{Nul}A) = 5 - 3 = 2$. The basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$ and the basis for Row$A$ is the set of non-zero **rows** of B: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$. To find the basis for Nul$A$, reduce the matrix B to reduced-echelon form to find the solutions to the trivial equation:

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

So the basis for Nul$A$ is: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Page 236, Problem 3:

For the same reasons problem 4, rank$A = 3$ and $\dim(\text{Nul}A) = 3$. The basis for Col$A$ is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ and the basis for Row$A$ is $\left\{ \begin{bmatrix} 2 \\ 6 \\ -6 \\ 6 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. Reducing B results in

$$\begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which implies } \mathbf{x} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the basis for Nul$A$ is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Page 237, Problem 7:

Because A is a 4×7 matrix, Col$A$ must be a subspace of $\mathbb{R}^4$. Since there are 4 pivot positions, it must be that Col$A = \mathbb{R}^4$.

Nul$A$ must be a three-dimensional subspace of $\mathbb{R}^7$ (the vectors in Nul$A$ have 7 entries). Therefore, Nul$A \neq \mathbb{R}^3$.

Page 237, Problem 8:

Because there are four pivot columns, dim(Col$A$) = 4, so dim(Nul$A$) $= 8 - 4 = 4$. It is impossible for Col$A = \mathbb{R}^4$ because Col$A$ is a subspace of $\mathbb{R}^6$ (the vectors in Col$A$ have 6 entries).

Page 237, Problem 9:

Because dim(Nul$A$) = 3 and $n = 6$, dim(Col$A$) $= 6 - 3 = 3$. It is impossible for Col$A = \mathbb{R}^3$ because Col$A$ is a subspace of $\mathbb{R}^4$ (the vectors in Col$A$ have 4 entries).

Page 237, Problem 11:

Because dim(Nul$A$) = 3 and $n = 5$, dim(Row$A$) = dim(Col$A$) $= 5 - 3 = 2$.

Page 237, Problem 18a:

True or False: If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

FALSE: As before, the pivot columns in B tell which columns of A form a basis for the column space of A.

Page 237, Problem 18c:

True or False: The dimension of the null space of A is the number of columns of A that are not pivot columns.

TRUE: Because the number of columns of A that are pivot columns equals the rank of A, by the Rank Theorem, the number of columns of A that are not pivot columns must be the dimension of the null space of A (see the proof of the Rank Theorem on page 233).

Page 238, Problem 31:

Compute $A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. Each column of this matrix is a multiple of $\mathbf{u}$, so dim(Col$A$) = 1, unless $a = b = c = 0$, in which case dim(Col$A$) = 0. Because dim(Col$A$) = rank$A$, rank$\mathbf{u}\mathbf{v}^{\mathrm{T}}$ = rank$A \leq 1$.

Page 238, Problem 32:

Notice that the second row of the matrix is twice the first. Therefore, take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, so that

$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$.