MATH 221, Spring 2018 - Homework 9 Solutions

Due Tuesday, April 24

Section 3.1

Page 168, Problem 15:

Using the diagonal product method results in:

$\det A = 3(3)(-1) + (0)(2)(0) + (4)(2)(5) - (0)(3)(4) - (5)(2)(3) - (-1)(2)(0) = -9 + 0 + 40 - 0 - 30 + 0 = 1$

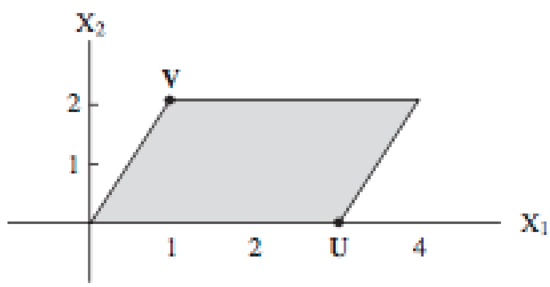
Page 168, Problem 30:

Use cofactors:

$\det A = 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} & 0 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1(-1) = -1$

Page 168, Problem 41:

The graph of the parallelogram is:



The formula for the area of a parallelgoram is $A = bh$, where b is the length of the base and h is height. From this picture, it is clear that the height is 2 and the base is 3, so the area is 6.

The determinant of $[\mathbf{u}\,\mathbf{v}]$ is equal to 6.

If the first entry of $\mathbf{v}$ is changed, the area of the parallelogram is still 6 and the determinant of the matrix is still 6.

Page 168, Problem 46:

$\det A^{-1} = \frac{1}{\det A}$

Section 3.2

Page 175, Problem 6:

First, row-reduce the matrix: $\begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$

The determinant of the last matrix is the product of the diagonal. However, in step 2, we multiplied row 2 by 1/6, so the determinant of the original matrix is the determinant of the last matrix multiplied by 6, or $6(1)(-3)(1) = -18$.

Page 175, Problem 15:

In one step, we see $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(7) = 35$

Page 175, Problem 17:

Row 2 and row 3 have been interchanged, so $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -1 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -1(7) = -7$

Page 175, Problem 19:

Note that the second row is first multiplied by two before the first row is added to it. Adding the first row to the second row has no impact on the determinant. However, the multiplication of the second row does matter. Therefore,

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$$

Page 175, Problem 22:

Compute the determinant (see below). Since the determinant is 0, the matrix not invertible.

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1 & -3 & -2 \\ 5 & 0 & -1 \\ 0 & 5 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1 & -3 & -2 \\ 0 & 15 & 9 \\ 0 & 5 & 3 \end{vmatrix} = -1(3) \begin{vmatrix} 1 & -3 & -2 \\ 0 & 5 & 3 \\ 0 & 5 & 3 \end{vmatrix} = -1(3) \begin{vmatrix} 1 & -3 & -2 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Page 175, Problem 25:

The set of vectors is linearly independent if and only if the determinant of the matrix formed by the vectors is not equal to 0. It can be shown that $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1$. Thus, the vectors are linearly independent.

Page 175, Problem 27a:

True or False: A row replacement operation does not affect the determinant of a matrix. **TRUE**

See Theorem 3.

Page 175, Problem 27c:

True or False: If the columns of A are linearly dependent, then $\det A = 0$. **TRUE**

By Theorem 4, A is invertible if and only if $\det A \neq 0$. Then, if A is not invertible (meaning the columns of A are linearly dependent) $\det A = 0$.

Page 175, Problem 27d:

True or False: $\det(A+B) = \det A + \det B$. **FALSE**

Any counterexample works, or the warning on page 173.

Section 5.1

Page 271, Problem 7:

4 is an eigenvalue if and only if the equation $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution, which is equivalent to solving the system

$(A-4I)\mathbf{x} = \mathbf{0}$: $(A-4I) = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$. Because the columns of this matrix

are linearly dependent, the system must have a nontrivial solution, so 4 is an eigenvalue. To find the eigenvector

corresponding to $\lambda = 4$, solve the system by row-reducing:

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Each vector of this form with $x_3 \neq 0$ is an eigenvector corresponding to $\lambda = 4$.

Page 271, Problem 9:

To find a basis for the eigenspace of each eigenvalue, find the vectors that span the eigenspace and are linearly

independent (i.e. the vectors that form the general solution of $(A - \lambda I)\mathbf{x} = \mathbf{0}$):

- When $\lambda = 1$: $A - I = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$. So, $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.
- When$\lambda = 3$: $A - 3I = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$. So, $\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.

Page 272, Problem 17:

Because the matrix is upper-triangular (every element below the diagnoal is 0), the eigenvalues are the entries of the

diagnoal. Thus, $\lambda = 0$, $\lambda = 3$, $\lambda = -2$.

Page 272, Problem 24:

Because the diagonal entries of an upper-triangular matrix are its eigenvalues, let $A = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$ where λ, $a \in \mathbb{R}$.

Thus, the diagonal entries are the eigenvalues, but because they are the same value, the matrix has one distinct eigenvalue.

Page 272, Problem 26:

Let $A^2 = 0$. Consider an arbitrary λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$. Then, consider

$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. Since $A^2 = 0$, it follows that $A^2\mathbf{x} = 0$, so $\lambda^2\mathbf{x}$. But we know $\mathbf{x} \neq 0$, so it must be that $\lambda^2 = 0$, which implies the only eigenvalue of A is 0.

Page 272, Problem 30:

Use the hint and prove exercises 27 and 29.

We begin by showing that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^t. We know λ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible. We want to show that λ is an eigenvalue of A^t if and only if $A^t - \lambda I$ is not invertible. But, we know (from the IMT) that $A - \lambda I$ is not invertible if and only if $(A - \lambda I)^t$ is not invertible. Since $(A - \lambda I)^t = A^t - (\lambda I)^t = A^t - \lambda I$, the result follows.

Next, we show that for an $n \times n$ matrix A with the property that the row sums all equal s, s must be an eigenvalue for A. Consider the vector $\mathbf{x} = \mathbf{1}$, where all entries are 1. Then, $A\mathbf{x} = \mathbf{s}$, where $\mathbf{s}$is a vector with all entries s (think of the definition of matrix multiplication). So, we have $A\mathbf{x} = \mathbf{s} = s\mathbf{x}$, where $\mathbf{x}$ is the vector of 1s.

To complete the original proof, we apply the above result to A^t and use the previously proved result involving eigenvalues of matrix transposes.