MATH 301
Homework 1 Answer Key

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Section 1.1

1. Let $A := \{ k \mid k \in \mathbb{N}, k \leqslant 20 \}$, $B := \{ 3k - 1 \mid k \in \mathbb{N} \}$, and $C := \{ 2k + 1 \mid k \in \mathbb{N} \}$. Determine the sets:

   (a) $A \cap B \cap C$,

   (b) $(A \cap B) \backslash C$,

   (c) $(A \cap C) \backslash B$.

   First give each set an explicit representation:

   (a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$,

   (b) $B = \{2, 5, 8, 11, 14, 17, 20, 23, 26, \ldots\}$,

   (c) $C = \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, \ldots\}$.

   Then we apply the set operations to get:

   (a) $A \cap B \cap C = \{ 5, 11, 17 \}$,

   (b) $(A \cap B) \backslash C = \{ 2, 8, 14, 20 \}$,

   (c) $(A \cap C) \backslash B = \{ 3, 7, 9, 13, 15, 19 \}$.

7. For each $n \in \mathbb{N}$, let $A_n = \{ (n + 1)k \mid k \in \mathbb{N} \}$.

   (a) What is $A_1 \cap A_2$?
   $A_1 = \{ 2k \mid k \in \mathbb{N} \}$ and $A_2 = \{ 3k \mid k \in \mathbb{N} \}$. Thus $A_1 \cap A_2 = \{ 6k \mid k \in \mathbb{N} \} = A_5$.

   (b) Determine the sets $\bigcup \{ A_n \mid n \in \mathbb{N} \}$ and $\bigcap \{ A_n \mid n \in \mathbb{N} \}$.
   $\bigcup \{ A_n \mid n \in \mathbb{N} \} = \mathbb{N} \backslash \{1\}$. For any natural number n greater than one we may consider the set A_{n-1} which must contain this n. However, there is no A_n containing one.

   $\bigcup \{ A_n \mid n \in \mathbb{N} \} = \varnothing$. For any natural number n the set A_n only contains elements larger than n. Thus n cannot be in the intersection over all $n \in \mathbb{N}$.

9. Let $A := B := \{ x \in \mathbb{R} \mid -1 \leqslant x \leqslant 1 \}$ and consider the subset $C := \{ (x, y) \mid x^2 + y^2 = 1 \}$ of $A \times B$. Is this set a function? Explain.

Proof. This set is not a function. If we consider $x = 0$, then we see that both $(0, 1)$ and $(0, -1)$ are in C. This violates the uniqueness conditions necessary for functions. ■

16. Show that the function f defined by $f(x) := x/\sqrt{x^2 + 1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{ y \mid -1 < y < 1 \}$.

Proof. First we check injectivity. Letting $x, z \in \mathbb{R}$ such that $f(x) = f(z)$ we have:

$$\begin{aligned}
\frac{x}{\sqrt{x^2+1}} &= \frac{z}{\sqrt{z^2+1}}, \\
\frac{x^2}{x^2+1} &= \frac{z^2}{z^2+1}, \\
\left(x^2\right)\left(z^2+1\right) &= \left(z^2\right)\left(x^2+1\right), \\
x^2z^2 + x^2 &= z^2x^2 + z^2, \\
x^2 &= z^2, \\
x &= \pm z.
\end{aligned}$$

Supposing that $x = -z$, we must have that $f(x) = -f(z)$. Thus we may conclude that $x = z$, showing injectivity.

Next we check surjectivity. Let $y \in (-1, 1)$ and consider $\frac{y}{\sqrt{1-y^2}}$. We note first that this value is real only when $y \in (-1, 1)$. Evaluating f at this point we have:

$$\begin{aligned}
f\left(\frac{y}{\sqrt{1-y^2}}\right) &= \\
\frac{\frac{y}{\sqrt{1-y^2}}}{\sqrt{\left(\frac{y}{\sqrt{1-y^2}}\right)^2 + 1}} &= \\
\frac{\frac{y}{\sqrt{1-y^2}}}{\sqrt{\frac{y^2}{1-y^2} + 1}} &= \\
\frac{y}{\sqrt{1-y^2}} \frac{1}{\sqrt{\frac{1}{1-y^2}}} &= \\
\frac{y}{\frac{\sqrt{1-y^2}}{\sqrt{1-y^2}}} &= y,
\end{aligned}$$

showing that f is surjective onto $(-1, 1)$. ■

Section 1.2

1. Prove that $\frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

   Proof. For $n = 1$ we have that $\frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$ proving our base case. Now suppose that our formula holds up to some natural number n. We then have:

$$\begin{aligned}\frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \\ \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} &= \\ \frac{n(n+2)+1}{(n+1)(n+2)} &= \\ \frac{n^2+2n+1}{(n+1)(n+2)} &= \\ \frac{(n+1)(n+1)}{(n+1)(n+2)} &= \frac{n+1}{n+2},\end{aligned}$$

   showing that our formula holds for $n+1$ as well. This completes the induction. ■

5. Prove that $1^2 - 2^2 + 3^2 + \cdots + (-1)^{n+1}n^2 = (-1)^{n+1}n(n+1)/2$ for all $n \in \mathbb{N}$.

   Proof. For $n = 1$ we have that $1^2 = 1 = 1 \cdot 1 \cdot 1 = (-1)^{1+1}1(1+1)/2$ proving our base case. Now suppose that our formula holds up to some natural number n. We then have:

$$\begin{aligned}1^2 + \cdots + (-1)^{n+1}n^2 + (-1)^{n+2}(n+1)^2 &= \\ (-1)^{n+1}\frac{n(n+1)}{2} + (-1)^{n+2}(n+1)^2 &= \\ (-1)^{n+1}\frac{n(n+1)}{2} + (-1)^{n+1}(-1)(n^2+2n+1) &= \\ (-1)^{n+1}\left(\frac{n(n+1)}{2} + (-1)(n^2+2n+1)\right) &= \\ (-1)^{n+1}\left(\frac{(n^2+n)+(-2)(n^2+2n+1)}{2}\right) &= \\ (-1)^{n+1}\left(\frac{n^2+n-2n^2-4n-2}{2}\right) &= \\ (-1)^{n+1}\left(\frac{-n^2-3n-2}{2}\right) &= \\ (-1)^{n+1}\left((-1)\frac{n^2+3n+2}{2}\right) &= (-1)^{n+2}\left(\frac{(n+1)(n+2)}{2}\right)\end{aligned}$$

   showing that our formula holds for $n+1$ as well. ■

9. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.

Proof. For $n = 1$ we have that $1^3 + (1+1)^3 + (1+2)^3 = 36 = 9 \cdot 4$ proving our base case. Now suppose that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for some natural number n. Concretely, let us say $9k = n^3 + (n+1)^3 + (n+2)^3$ for some natural number k. We then have:

$$\begin{aligned} (n+1)^3 + (n+2)^3 + (n+3)^3 &= \\ (n+1)^3 + (n+2)^3 + (n^3 + 9n^2 + 27n + 27) &= \\ n^3 + (n+1)^3 + (n+2)^3 + 9(n^2 + 3n + 3) &= \\ 9k + 9(n^2 + 3n + 3) &= 9(k + n^2 + 3n + 3). \end{aligned}$$

This shows that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is divisible by 9, completing the induction. ■

18. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for all $n \in \mathbb{N}$, $n > 1$.

Proof. For $n = 2$ we have that:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} = \sqrt{2} + \frac{2 - \sqrt{2}}{2} > \sqrt{2},$$

proving our base case. Now suppose that this inequality holds up to some natural number n. We then have:

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} > \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1},$$

showing that our formula holds for $n + 1$ as well. ■

19. Let S be a subset of $\mathbb{N}$ such that (a) $2^k \in S$ for all $k \in \mathbb{N}$, and (b) if $k \in S$ and $k \geqslant 2$, then $k - 1 \in S$. Prove that $S = \mathbb{N}$.

Proof. First let n be some positive natural number, then we have that $n < 2^n$ and $n + 1 \geqslant 2$. We may use clause (a) to say that $2^n \in S$ and then repeatedly apply clause (b) to say that $n \in S$. Thus $S = \mathbb{N}$. ■