MATH 301
Homework 3 Answer Key

Matthew Kousoulas

Section 2.1

4. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.

   Proof. Suppose that a is nonzero. We may then invoke theorem 2.1.2 to say that $a = 1$. ■

6. Use the argument in the proof of Theorem 2.1.4 to show that there does not exist a rational number s such that $s^2 = 6$.

   Proof. Suppose that there exist integers p and q with no nontrivial common factors such that $(p/q)^2 = 6$. Since $p^2 = 6q^2$ we see that p^2 is a multiple of both 2 and 3. This implies that p is also a multiple of 2 and 3 since these are both prime numbers. (If a prime number divides the product of two numbers, then it must divide at least one of those two numbers.) Since p is a multiple of 2 and 3 there is some natural number $m \in \mathbb{N}$ such that $p = 6m$ and thus $36m^2 = 6q^2$ so that $6m^2 = q^2$. As above this implies that q is a multiple of 2 and 3, contradicting that q and p have no common factors. Thus we may conclude that no such p and q exist. ■

9. Let $K := \{\, s + t\sqrt{2} \mid s, t \in \mathbb{Q} \,\}$. Show that K satisfies the following:

   (a) If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.

   Proof. By assumption there exist $p_1, p_2, q_1, q_2 \in \mathbb{Q}$ such that $x_1 = p_1 + q_1\sqrt{2}$ and $x_2 = p_2 + q_2\sqrt{2}$. We thus have the following:

   $$x_1 + x_2 = p_1 + q_1\sqrt{2} + p_2 + q_2\sqrt{2} = (p_1 + p_2) + (q_1 + q_2)\sqrt{2},$$

   and

   $$x_1 x_2 = (p_1 + q_1\sqrt{2})(p_2 + q_2\sqrt{2}) = (p_1 p_2 + 2q_1 q_2) + (p_1 q_2 + p_2 q_1)\sqrt{2}.$$

   Since $\mathbb{Q}$ is a closed under multiplication and addition these are both elements of K. ■

(b) If $x \neq 0$ and $x \in K$, then $1/x \in K$.

Proof. As above, let $x = p + q\sqrt{2}$. We then have:

$$\frac{1}{x} = \frac{1}{p + q\sqrt{2}} = \frac{1}{p + q\sqrt{2}} \cdot \frac{p - q\sqrt{2}}{p - q\sqrt{2}} = \frac{p - q\sqrt{2}}{p^2 + 2q^2} = \frac{p}{p^2 + 2q^2} - \frac{q}{p^2 + 2q^2}\sqrt{2},$$

which is also an element of K. ■

20. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.

Proof. We note that $c \neq 0$ since $0 < c$, and thus $0 < c^2$ by theorem 2.1.8. We may then multiply $c < 1$ by c to get $c^2 < c$ by theorem 2.1.7. Combining all these inequalities with the same theorem lets us say $0 < c^2 < c < 1$. ■

(b) If $1 < c$, show that $1 < c < c^2$.

Proof. We may use theorem 2.1.7 similarly the above to say that $1 < c$ implies $c < c^2$ and thus $1 < c < c^2$. ■

21. (a) Prove there is no $n \in \mathbb{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathbb{N}$.)

Proof. Let S be the set of all $n \in \mathbb{N}$ such that $0 < n < 1$. By well-ordering, if this set is nonempty then it must have a least element. Let us call this element s. We then have that $0 < s < 1$ implies $0 < s^2 < s < 1$ by the above result. Since s is a natural number so is s^2. However, this contradicts our assumption that s was the least element of S. Thus we may conclude that S is empty. ■

(b) Prove that no natural number can be both even and odd.

Proof. A number k is even if there is some natural number n such that $k = 2n$ and odd if there is a natural number n' such that $k = 2n' + 1$. Supposing k is both even and odd we must have the existence of natural numbers n and n' such that $2n = 2n' + 1$ or equivalently $2(n - n') = 1$ and thus $0 < n - n' < 1$. By the above proof this cannot happen. ■

23. If $a > 0, b > 0$, and $n \in \mathbb{N}$, show that $a < b$ if and only if $a^n < b^n$.

Proof. We note that this is immediate in the case that n is one. Now suppose that $a < b$ if and only if $a^n < b^n$ for some n. Suppose first that $a < b$. This implies $a^n < b^n$ by the above and further that $a \cdot a^n < a \cdot b^n$ and $a \cdot b^n < b \cdot b^n$ which give us $a^{n+1} < b^{n+1}$. Conversely, suppose that $b \leqslant a$. Our inductive hypothesis states that this implies $b^n \leqslant a^n$ and thus $b^{n+1} \leqslant a^{n+1}$ by the same logic as above. This completes the inductive step, and we may conclude that the proposition holds for all n. ■

Section 2.2

3. If $x, y, z \in \mathbb{R}$ and $x \leqslant z$, show that $x \leqslant y \leqslant z$ if and only if $|x-y|+|y-z|=|x-z|$. Interpret this geometrically.

   Proof. Suppose $x \leqslant y \leqslant z$. We thus have that $|x-y|=y-x$, $|y-z|=z-y$, and $|x-z|=z-x$. Therefore:

   $$|x-y|+|y-z|=y-x+z-y=z-x=|x-z|.$$

   Conversely, since $x \leqslant z$ we have $|x-z|=z-x$, and thus $|x-y|+|y-z|=z-x$. We consider three overlapping cases: $y \leqslant x$, $x \leqslant y \leqslant z$, and $z \leqslant y$. Supposing $y \leqslant x$, we would then have $x-y+z-y=z-x$ implying $x=y$. Similarly, $z \leqslant y$ will imply that $z=y$. We thus have that $x \leqslant y \leqslant z$ in every case.

   This makes sense geometrically since absolute value is a measure of the line segments connecting two points. When y is between x and z the line segment from x to z is precisely the concatenation of the line segments from x to y and y to x. (Allowing of course for the possibility that a line segment is merely a point.) ■

5. If $a<x<b$ and $a<y<b$, show that $|x-y|<b-a$. Interpret this geometrically.

   Proof. From the second equation it follows that $-b<-y<-a$. We may then subtract these two inequalities to get $-(b-a)=a-b<x-y<b-a$. We may then apply theorem 2.2.2 to say that $|x-y|<b-a$.

   This result can be interpreted to mean that given any two points inside a finite interval, the length of the segment joining those two points cannot be longer than the length of the interval. ■

7. Find all $x \in \mathbb{R}$ that satisfy the equation $|x+1|+|x-2|=7$.

   We consider three intervals: $(-\infty,-1]$, $(-1,2]$, and $(2,\infty)$. In the first we have $(-x-1)+(2-x)=7 \implies x=-3$, in the second we have $(x+1)+(2-x)=7$, which has no solution, and in the third we have $(x+1)+(x-2)=7 \implies x=4$. Thus both -3 and 4 are solutions.

12. Find all $x \in \mathbb{R}$ that satisfy the inequality $4<|x+2|+|x-1|<5$.

    We consider three intervals: $(-\infty,-2]$, $(-2,1]$, and $(1,\infty)$. In the first we have $4<(-x-2)+(1-x)<5 \implies -3<x<-5/2$, in the second we have $4<(x+2)+(1-x)<5$, which has no solution, and in the third we have $4<(x+2)+(x-1)<5 \implies 3/2<x<2$. This gives a final solution set of $(-3,-5/2)\cup(3/2,2)$.

14. Determine and sketch the set of pairs (x,y) in $\mathbb{R}\times\mathbb{R}$ that satisfy:

    (a) $|x|=|y|$: The solution set here is the lines $x=y$ and $x=-y$, i.e. $\{(x,y)\in\mathbb{R}^2 \mid x=\pm y\}$.

    (b) $|x|+|y|=1$: There are four cases based on the sign of x and y. In each we get a line segment connecting two of the points in $(1,0),(0,1),(-1,0),(0,-1)$ in a cycle.

(c) $|xy| = 2$: We have the two cases where x and y have matching or opposite signs, giving the hyperbolas $y = 2/x$ and $y = -2/x$ respectively.

(d) $|x| - |y| = 2$: This is similar to the first case. The solution is clear if we do a case analysis on the sign of x to get $x = |y| + 2$ and $x = -|y| - 2$ when x is positive and negative respectively.

16. Let $\varepsilon > 0$ and $\delta > 0$, and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.

Proof. Let us first fix values $\varepsilon, \delta > 0$ and $a \in \mathbb{R}$. Unpacking the definition of neighborhoods gives us $V_\varepsilon(a) := \{\, x \in \mathbb{R} \mid |x - a| < \varepsilon \,\}$ and likewise for δ. We then have:

$$V_\varepsilon(a) \cup V_\delta(a) = \{\, x \in \mathbb{R} \mid |x - a| < \varepsilon \text{ or } |x - a| < \delta \,\},$$

and:

$$V_\varepsilon(a) \cap V_\delta(a) = \{\, x \in \mathbb{R} \mid |x - a| < \varepsilon \text{ and } |x - a| < \delta \,\}.$$

Letting $\gamma_1 = \max\{\varepsilon, \delta\}$ and $\gamma_2 = \min\{\varepsilon, \delta\}$ it is clear that:

$$V_\varepsilon(a) \cup V_\delta(a) - \{\, x \in \mathbb{R} \mid |x - a| < \gamma_1 \,\} := V_{\gamma_1}(a),$$

and:

$$V_\varepsilon(a) \cap V_\delta(a) - \{\, x \in \mathbb{R} \mid |x - a| < \gamma_2 \,\} := V_{\gamma_2}(a).$$

■

19. Show that if $a, b, c \in \mathbb{R}$, then the "middle number" is:

$$\operatorname{mid}\{a, b, c\} = \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}.$$

Explanation. Without loss of generality suppose that $a \leqslant b \leqslant c$. (If it were otherwise, we could simply relabel our variables.) We then have $\max\{a, b\} = b$, $\max\{b, c\} = c$, and $\max\{c, a\} = c$. Further $\min\{b, c, c\} = b$. This conforms with what we think the "middle number" should be. We note that by using weak inequality we avoid the need to explicitly consider edge cases. You should prove to yourself that the above explanation still makes sense when any of the variables are equal. ■