MATH 301
Homework 4 Answer Key

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Section 2.3

3. Let $S_3 := \{1/n | n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 \geqslant 0$.

   Proof. For $\sup S_3 = 1$, we first note that $1 \geqslant 1/n$ for all $n \in \mathbb{N}$, and thus 1 is an upper bound of S_3. Since $1 \in S_3$ this proves that $\sup S_3 = 1$.

   For $\inf S_3 \geqslant 0$ we merely note that $1/n$ is positive for all n and consequently 0 is a lower bound of S_3. It follows immediately that the infimum of S_3 is greater than or equal to 0. ■

4. Let $S_4 := \{1 - (-1)^n/n | n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.

   Proof. We note that $1 - (-1)^1/1 = 2$ and $1 - (-1)^2/2 = 1/2$ are both in S_4. Further, $s \leqslant 2$ for any $s \in S_4$ since $|(-1)^n/n| = 1/n \leqslant 1$. Thus 2 is an upper bound of S_4 and since it is also contained in S_4 it must be the supremum. Similarly, for all n greater than one we have $|(-1)^n/n| = 1/n \leqslant 1/2$ and consequently that $s \geqslant 1/2$ for all $s \in S_4$. Thus $1/2$ is a lower bound and further the infimum of S_4. ■

5c. Find the infimum and supremum, if they exist, of the set:

$$C := \{x \in \mathbb{R} | x < 1/x\}.$$

   Proof. We note that for any negative number x less than 1 $x < 1/x$ and so it follows $\inf C$ does exist. Similarly for any positive number x greater than or equal to 1 $x \geqslant 1/x$ and so all such x are upper bounds of C. We note further $x < 1/x$ for any x in $(0, 1)$. Thus if we take any $x < 1$ we have that $x + (1 - x)/2 < 1$ and consequently in S_4. This shows that no $x < 1$ can be an upper bound of S_4 and further that 1 is the supremum. ■

6. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s | s \in S\}$.

   Proof. Let w be the infimum of S. This means that $w \leqslant s$ for all $s \in S$ and for all $w' \in \mathbb{R}$ such that $w' \leqslant s$ for all $s \in S$ we have $w' \leqslant w$. We may multiply all of the above inequalities by negative one to get that $-w \geqslant -s$ for all $s \in S$ and if $-w' \geqslant -s$ for all $s \in S$ then $-w' \geqslant -w$. This is precisely the statement that $-w$ is the supremum of $\{-s | s \in S\}$. ■

7. If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.

   Proof. Let $u \in S$ be an upper bound of S and $u' \in \mathbb{R}$ be another not necessarily distinct upper bound of S. Since $u \in S$ we immediately have that $u' \geqslant u$ and thus that u is the supremum of S. ∎

8. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply $t \notin S$.

   Proof. Suppose that u is an upper bound of S. Since $u \geqslant s$ for all $s \in S$ we have immediately that $u < t$ implies $t \notin S$. Conversely suppose we have an element $u \in \mathbb{R}$ such that if $u < t$ then $t \notin S$ for all $t \in \mathbb{R}$. The contrapositive gives us that if $s \in S$ then $s \leqslant u$ implying that u is an upper bound of S. ∎

11. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that $\inf S \leqslant \inf S_0 \leqslant \sup S_0 \leqslant \sup S$.

   Proof. Given $s \in S_0$ we have $s \in S$ and consequently $\inf S \leqslant s$ implying that $\inf S$ is a lower bound of S_0 and that $\inf S \leqslant \inf S_0$. An analogous argument shows that $\sup S_0 \leqslant \sup S$. Since S_0 is nonempty we may take $s \in S_0$. (We note that the above proof did not explicitly require that S or S_0 be nonempty. The above inequalities will hold if S_0 is empty, however the following will not unless S is also empty.) We then have $\inf S_0 \leqslant s \leqslant \sup S_0$. The inequality we wish to show is then a result of using the transitive property several times. ∎

14. Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon > 0$ there exists $t \in S$ such that $t < w + \varepsilon$.

   Proof. Let w be the infimum of S and $\varepsilon > 0$, then $w < w + \varepsilon$ implying that $w + \varepsilon$ is not a lower bound of S and thus that there is some $t \in S$ such that $t < w + \varepsilon$. Conversely, let w and w' be lower bounds of S such that $w \leqslant w'$. We then have that there is some $\varepsilon \geqslant 0$ such that $w + \varepsilon = w'$. Since w' is a lower bound, it cannot be the case that $\varepsilon > 0$ otherwise we would have that there is some t greater than w' contradicting that it is a lower bound. We are thus forced to conclude that $\varepsilon = 0$ implying that $w = w'$ and that w is the infimum of S. ∎

Section 2.4

3. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + 1/n$ is an upper bound of S, then $u = \sup S$.

   Proof. Suppose we have some $s \in \mathbb{R}$ such that $u < s$, we then have the existence of some $\varepsilon > 0$ such that $u + \varepsilon = s$. Further the Archimedean property gives us n such that $u + 1/n < u + \varepsilon$. Since $u + 1/n$ is an upper bound of S it must be the case that $s \notin S$. This is equivalent to saying that given an $s \in S$ it must be the case that $s \leqslant u$ and thus that u is an upper bound of S.

   Further, given an $\varepsilon > 0$ we have an $n \in \mathbb{N}$ such that $1/n < \varepsilon$ and consequently $u - \varepsilon < u - 1/n$ and thus that there exists some $s \in S$ such that $u - \varepsilon < s$ since $u - 1/n$ is not an upper bound of S. Since u is an upper bound this is equivalent to saying that u is the supremum of S. ∎

4. Let S be a nonempty bounded set in $\mathbb{R}$.

   (a) Let $a > 0$, and let $aS := \{as | s \in S\}$. Prove that:

   $$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$

   Proof. Let $w := \sup S$. We have $w \geqslant s$ for all $s \in S$ and if $w' \geqslant s$ for all $s \in S$ then $w' \geqslant w$. Since a is positive we may multiple through each of these inequalities to say respectively that aw is an upper bound of aS and if aw' is an upper bound of aS then $aw' \geqslant aw$ implying that aw is the supremum of aS.

   An analogous argument will prove the statement about infima. ∎

   (b) Let $b < 0$, and let $bS := \{bs | s \in S\}$. Prove that:

   $$\inf(bS) = a \sup S, \quad \sup(bS) = b \inf S.$$

   Proof. This proof proceeds as above, however this time multiplication by b reverses the direction of each inequality. Thus a statement about suprema becomes a statement about infima and vice versa. ∎

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by:

$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geqslant y. \end{cases}$$

   (a) $f(x) := \sup\{h(x, y) | y \in Y\} = 1$ since for any x we may find a y such that $x \leqslant y$, and consequently $\inf\{f(x) | x \in X\} = 1$.

   (b) $g(x) := \inf\{h(x, y) | x \in X\} = 0$ since for any y we may find a x such that $x < y$, and consequently $\sup\{g(x) | x \in X\} = 0$.

14. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.

Proof. By a corollary to the Archimedean property there exists an $n \in \mathbb{N}$ such that $1/n < y$. Further $1/2^n < 1/n$ since $n < 2^n$. ■

17. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number u such that $u^3 = 2$.

Proof. Analogously to the proof in the book we define $S := \{s \in \mathbb{R} | 0 \leqslant s, s^3\}$. The set is again bounded above by 2 so we may apply the supremum property to say that S has supremum $x \in \mathbb{R}$. Supposing $x^3 < 2$ we may choose n so that:

$$\frac{1}{n} < \frac{2 - x^3}{3x^2 + 3x + 1}, \quad \text{implying:} \quad \frac{1}{n}\left(3x^2 + 3x + 1\right) < 2 - x^3,$$

and thus:

$$\left(x + \frac{1}{n}\right)^3 = x^3 + \frac{3x^2}{n} + \frac{3x}{n^2} + \frac{1}{n^3} < x^3 + \frac{1}{n}\left(3x^2 + 3x + 1\right) < 2,$$

contradicting that x is the supremum.

Supposing instead that $x^3 > 2$ we may choose m so that:

$$\frac{1}{m} < \frac{x^3 - 2}{3x^2}, \quad \text{implying:} \quad \frac{1}{m} 3x^2 < x^3 - 2,$$

and thus:

$$\left(x - \frac{1}{m}\right)^3 = x^3 - \frac{3x^2}{m} + \frac{3x}{m^2} - \frac{1}{m^3} > x^3 - \frac{3x^2}{m} > 2,$$

again contradicting that x is the supremum.

We may thus conclude that $x^3 = 2$. ■

19. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru | r \in Q\}$ is dense in $\mathbb{R}$.)

Proof. The case where $u = 1$ was shown in the book. From that theorem we have some r' such that $x < r' < y$. Letting $r = r'/u$ we then have $x < ru < y$. ■