MATH 301
Homework 5 Answer Key

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Section 2.5

7. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

   Proof. We note first that $0 \in [0, 1/n]$ for all $n \in \mathbb{N}$ implying that $\{0\} \subseteq \bigcap_{n=1}^{\infty} I_n$. Conversely let $a \in \bigcap_{n=1}^{\infty} I_n$. This implies that $a \in [0, 1/n]$ for all $n \in \mathbb{N}$ by the definition of intersections. By the definition of intervals we then have that $0 \leqslant a \leqslant 1/n$ for all $n \in \mathbb{N}$. Clearly a cannot be less than zero. Supposing on the other hand that a were greater than zero then there would be some $n \in \mathbb{N}$ such that $1/n < a$ by corollary to the Archimedean property; thus a must be zero. This implies that $\bigcap_{n=1}^{\infty} I_n \subseteq \{0\}$. We may thus conclude $\bigcap_{n=1}^{\infty} I_n = \{0\}$. ■

8. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \varnothing$.

   Proof. Considering an element $a \in \bigcap_{n=1}^{\infty} J_n$, we must have $a \in (0, 1/n)$ as before. However, this time the inequality is strict giving us that $0 < a < 1/n$ for all $n \in \mathbb{N}$. Since no real number can satisfy this (using reasoning analogous to the above problem) we must conclude that a does not exist and hence $\bigcap_{n=1}^{\infty} J_n$ is empty. Since the empty set is unique, this is all we needed to show. ■

9. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n = \varnothing$.

   Proof. We proceed as before. Let $a \in \bigcap_{n=1}^{\infty} K_n$. It must be the case that $n < a$ for all $n \in \mathbb{N}$. However, this is a clear violation of the Archimedean property which states that for any real number there is some natural number greater than it. We again conclude that a cannot exist as such and thus that $\bigcap_{n=1}^{\infty} K_n$ is empty. ■

14. Show that if $a_k, b_k \in \{0, 1, \ldots, 9\}$ and if:

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^m} \neq 0,$$

   then $n = m$ and $a_k = b_k$ for $k = 1, \ldots, n$.

   Proof. We may assume that $a_n \neq 0$. If $n > m$, then multiply by 10^n to get $10p + a_n = 10q$ where $p, q \in \mathbb{N}$, so that $a_n = 10(q - p)$. Since $q - p \in \mathbb{Z}$ while a_n is one of the digits $0, 1, \ldots, 9$, it follows that $a_n = 0$, a contradiction. Therefore $n \leqslant m$, and a similar argument shows that $m \leqslant n$; therefore $n = m$.

   Repeating the above argument with $n = m$, we obtain $10p + a_n = 10q + b_n$, so that $a_n - b_n = 10(q - p)$, whence it follows that $a_n = b_n$. If this argument is repeated, we conclude that $a_k = b_k$ for $k = 1, \ldots, n$. ■

17. What rationals are represented by the periodic decimals:

$$1.25137\cdots137\cdots \quad \text{and} \quad 35.14653\cdots653\cdots?$$

$$31253/24975 \quad \text{and} \quad 3511139/99900 \quad \text{respectively.}$$

Section 3.1

5. Use the definition of the limit of a sequence to establish the following limits.

   (a) $\lim\left(\dfrac{n}{n^2+1}\right) = 0$:

   Proof. Letting $\varepsilon > 0$ be arbitrary, we define $K(\varepsilon) \geqslant 1/\varepsilon$. (Any natural number greater than $1/\varepsilon$ will suffice, and the Archimedean property guarantees that at least one exists.) Then if $n \geqslant K(\varepsilon)$, $n \geqslant 1/ep$ and consequently:

   $$0 < \frac{n}{n^2+1} < \frac{n}{n^2} = 1/n < \varepsilon.$$

   ■

   (b) $\lim\left(\dfrac{2n}{n+1}\right) = 2$:

   Proof. Letting $\varepsilon > 0$ put $K \geqslant 2/\varepsilon$. Then $n > K$ implies $n > 2/\varepsilon$ and thus $|2n/(n+1) - 2| = 2/(n+1) < 2/n < \varepsilon$. ■

   (c) $\lim\left(\dfrac{3n+1}{2n+5}\right) = \dfrac{3}{2}$:

   Proof. Letting $\varepsilon > 0$ put $K \geqslant 13/4\varepsilon$. Then $n > K$ implies $n > 13/4\varepsilon$ and thus $|(3n+1)/(2n+5) - 3/2| = 13/(4n+10) < 13/4n < \varepsilon$. ■

   (d) $\lim\left(\dfrac{n^2-1}{2n^2+3}\right) = \dfrac{1}{2}$:

   Proof. Letting $\varepsilon > 0$ put $K \geqslant 5/4\varepsilon$. Then $n > K$ implies $n > 5/4\varepsilon$ and thus $|(n^2-1)/(2n^2+3) - 1/2| = 5/(4n^2+6) < 5/4n^2 \leqslant 5/4n < \varepsilon$. ■

7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.

   (a) Use the definition of limit to show that $\lim(x_n) = 0$.

   Proof. Letting $\varepsilon > 0$ put $K \geqslant e^{1/\varepsilon} - 1$. Then $\ln(n+1) \geqslant 1/\varepsilon$ and consequently $0 < 1/\ln(n+1) \leqslant \varepsilon$ for all $n > K$. Therefore $\lim(x_n) = 0$. ■

   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.

   For $\varepsilon = 1/2$, $e^{1/\varepsilon} \approx 6.3$ so let $K(1/2) = 7$, and for $\varepsilon = 1/10$, $e^{1/\varepsilon} \approx 22025.5$ so let $K(1/10) = 22026$.

8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

   Proof. This follows from the fact that $||x_n|| = |x_n|$. For any $\varepsilon > 0$ we have that $||x_n|| < \varepsilon$ if and only if $|x_n| < \varepsilon$. However, if $(|x_n|)$ converges to a value other than zero, say as in the sequence $(|(-1)^n|)$, which converges to one, it is not the case that $((-1)^n)$ also converges. ■

10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geqslant M$.

Proof. Let $\varepsilon = x/2$, by the definition of limit there exists an $M \in \mathbb{N}$ such that $|x_n - x| < x/2$ for all $n \geqslant M$. Thus $-x/2 < x_n - x < x/2$ or $0 < x/2 < x_n$ for all such n. ■

17. Show that $\lim(2^n/n!) = 0$.

Proof. We note that:

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2}{1 \cdot 2 \cdot 3 \cdots n} = 2 \cdot 1 \cdot \frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n} \leqslant 2 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdots \frac{2}{3} = 2\left(\frac{2}{3}\right)^{n-2}.$$

Further,

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2}{1 \cdot 2 \cdot 3 \cdots n} \geqslant \frac{2 \cdot 2 \cdot 2 \cdots 2}{1 \cdot 2 \cdot 3 \cdots n} \cdot \frac{2}{n+1} = \frac{2^{n+1}}{(n+1)!}.$$

Therefore, letting $\varepsilon > 0$ we may define $K \geqslant \log_{2/3}(\varepsilon/2) + 2$ to get $2^n/n! \leqslant \varepsilon$ for all $n > K$. ■

18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geqslant K$, then $\frac{1}{2}x < x_n < 2x$.

Proof. By taking $\varepsilon := \min\{x - \frac{1}{2}x, 2x - x\}$ (this value is positive since x is positive) there must exist a K such that $|x_n - x| < \varepsilon$ implying:

$$\frac{1}{2}x - x \leqslant -\varepsilon < x_n - x < \varepsilon \leqslant 2x - x,$$

and consequently $\frac{1}{2}x < x_n < 2x$ for all $n \geqslant K$. ■