MATH 301
Homework 6 Answer Key

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Section 3.2

6. Find the limits of the following sequences:

   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$:

   Proof. First we rewrite:

$$\frac{\sqrt{n}-1}{\sqrt{n}+1}\cdot\frac{\sqrt{n}}{\sqrt{n}} = \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}.$$

   Then we may decompose the limit as follows:

$$\lim\left(\frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}\right) = \frac{1-\sqrt{\lim(1/n)}}{1+\sqrt{\lim(1/n)}} = 1.$$

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   (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$:

   Proof. First we rewrite:

$$\frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}}.$$

   Then we may decompose the limit as follows:

$$\lim\left(\frac{1}{\sqrt{n}}+\frac{1}{n\sqrt{n}}\right) = \sqrt{\lim\left(\frac{1}{n}\right)} + \lim\left(\frac{1}{n}\right)\sqrt{\lim\left(\frac{1}{n}\right)} = 0.$$

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7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot* be used.

   Proof. There exists some M such that $b_n < M$ for all n and for each $\varepsilon > 0$ there exists a K such that $a_n < \varepsilon/M$ for all $n > K$. Thus $a_n b_n < a_n M < \varepsilon$ for all such n.

   We cannot use Theorem 3.2.3 because we are not given that (b_n) converges. ■

9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.

Proof. We may rewrite:

$$y_n = \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

and thus:

$$\sqrt{n}y_n = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + 1/n} + 1}.$$

We may then decompose the limit:

$$\lim\left(\sqrt{n}y_n\right) = \frac{1}{\sqrt{1 + \lim(1/n)} + 1} = \frac{1}{2}.$$

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12. If $0 < a < b$, determine $\lim\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$.

Proof. We have:

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a(a/b)^n + b}{(a/b)^n + 1} \cdot \frac{b^n}{b^n}.$$

Further:

$$\lim\left(\frac{a(a/b)^n + b}{(a/b)^n + 1}\right) = \frac{a\lim((a/b)^n) + b}{\lim((a/b)^n) + 1} = b,$$

since $a < b$ and thus $\lim((a/b)^n) = 0$. ■

15. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

Proof. We have $b^n \leqslant (a^n + b^n) \leqslant 2b^n$ since $0 < a < b$, and thus $b \leqslant z_n \leqslant 2^{1/n}b$. Because $2^{1/n}$ converges to 1 (put $K(\varepsilon) > 1/\log_2(\varepsilon + 1)$) we thus have z_n converges to 1 by the squeeze theorem. ■

22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geqslant M$. Does it follow that (y_n) is convergent?

Proof. Let $x := \lim(x_n)$, then for all $\varepsilon > 0$ there exists a K such that $|x - x_n| < \varepsilon/2$ for all $n > K$. Similarly we have an M such that $|x_n - y_n| < \varepsilon/2$ for all $n > M$. Letting $N = \max\{K, M\}$ we have $|x - y_n| \leqslant |x - x_n| + |x_n - y_n| < \varepsilon$ for all $n > N$. Thus (y_n) converges to x. ■

23. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

Proof. By results from chapter 2 way me write:

$$u_n = 1/2(x_n + y_n + |x_n - y_n|) \quad \text{and} \quad v_n = 1/2(x_n + y_n - |x_n - y_n|).$$

We may then decompose these by theorems 3.2.3 and 3.2.9 to get:

$$\lim(u_n) = 1/2(\lim(x_n) + \lim(y_n) + |\lim(x_n) - \lim(y_n)|),$$

and

$$\lim(v_n) = 1/2(\lim(x_n) + \lim(y_n) - |\lim(x_n) - \lim(y_n)|).$$

Since $\lim(x_n$ and $\lim(y_n)$ exist, so must $\lim(u_n)$ and $\lim(v_n)$. ■