MATH 301
Homework 7 Answer Key

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Section 3.3

4. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

   Proof. Note first that $x_2 = \sqrt{3}$ and thus $x_1 < x_2$. Further, if $x_{n+1} - x_n > 0$, then $x_{n+2} - x_{n+1} = (x_{n+1} - x_n)/(\sqrt{2 + x_{n+1}} + \sqrt{2 + x_n}) > 0$ so (x_n) is increasing by induction. Similarly, $x_1 < 2$, and if $x_n < 2$, then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2$, so (x_n) is bounded by induction. This shows that (x_n) is convergent.

   To find the limit we note that the limit of (x_n) must equal the limit of its tails. Thus $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{2 + \lim_{n\to\infty} x_n}$ and the limit we are looking for is the fixed point of $f(x) = \sqrt{2 + x}$ in the interval $(1, 2]$ which is just 2. ■

8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leqslant \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.

   Proof. Note that both sequences are bounded by b_1 and a_1. Since both sequences are monotone they must be convergent. Finally, $\lim(b_n) - \lim(a_n) \geqslant 0$ because $a_n - b_n \geqslant 0$ for all $n \in \mathbb{N}$, and thus $\lim(a_n) \leqslant \lim(b_n)$.

   Noting that intervals can be fully described by their endpoints, a sequence of nested intervals gives two monotone sequences corresponding to the upper and lower bounds, whence the nested interval property. ■

9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.

   Proof. One of the definitions of supremum holds that for all $\varepsilon > 0$ there is $a \in A$ such that $u - \varepsilon < a \leqslant u$. Considering the sequence $(1/n)$ and the corresponding sequence (a_n) we have $u - (1/n) < a_n \leqslant u$. By the squeeze theorem $a_n \to u$. ■

11. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges.

    Proof. Note that $x_1 = 1 < 5/4 = x_2$ and that $x_n < x_n + 1/(n+1)^2 = x_{n+1}$ and thus (x_n) is increasing by induction. Further, noting $1/n^2 < 1/(n-1)n$ we have:

    $$x_n < 1 + 1/2 + 1/6 + \cdots + 1/(n-1)n = 1 + (1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-1) - 1/n) = 2 - 1/n < 2,$$

    and thus (x_n) is bounded above and hence converges. ■

12. Establish the convergence and find the limits of the following sequences.

(a) $\left((1 + 1/n)^{n+1}\right)$,

$$(1 + 1/n)^{n+1} = (1 + 1/n)^n(1 + /1/n) \to e \cdot 1 = e$$

(b) $\left((1 + 1/n)^{2n}\right)$,

$$(1 + 1/n)^{2n} = ((1 + 1/n)^n)^2 \to e^2$$

(c) $\left((1 + 1/(n + 1))^n\right)$,

$$(1 + 1/(n + 1))^n = (1 + 1/(n + 1))^{n+1}/(1 + 1/(n + 1)) \to e/1 = e$$

(d) $\left((1 - 1/n)^n\right)$,

$$(1 - 1/n)^n = (1 + 1/(n - 1))^{-n} \to e^{-1}$$

Section 3.4

3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

Proof. Note that (f_n) is increasing so $x_n \geqslant 1$ for all $n \in \mathbb{N}$ implying $L > 0$. Further $x_n = f_{n+1}/f_n = (f_{n-1} + f_n)/f_n = 1/x_{n-1} + 1$ and so $L = 1/L + 1$. Solving gives $L = \frac{1}{2}(1 + \sqrt{5})$. ■

6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.

(a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \geqslant 3$. Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.

Proof. We have the following are equivalent:

$$\begin{aligned} x_{n+1} &< x_n \\ (n + 1)^{1/(n+1)} &< n^{1/n} \\ (n + 1)^n &< n^{n+1} \\ (1 + 1/n)^n &< n. \end{aligned}$$

Note from example 3.3.6 $(1 + 1/n)^n < e$ and $e < n$ when $n \geqslant 3$. Thus x_n has a tail that is monotonically decreasing and bounded below and thus has a limit. ■

(b) Use the fact that the subsequence (x_{2_n}) also converges to x to conclude that $x = 1$.

Proof. Let $x := \lim(x_n)$, then

$$x = \lim(x_{2n}) = \lim\left((2n)^{1/2n}\right) = \lim\left(\left(2^{1/n}n^{1/n}\right)^{1/2}\right) = x^{1/2},$$

which has solutions $x = 0$ and $x = 1$. Since $x_n \geqslant 1$ for all n it must be that $x = 1$. ■

8. Determine the limits of the following.

   (a) $((3n)^{1/2n})$,
   $$(3n)^{1/2n} = ((3n)^{1/3n})^{3/2} \to 1^{2/3} = 1,$$

   (b) $((1 + 1/2n)^{3n})$,
   $$((1 + 1/2n)^{3n}) = ((1 + 1/2n)^{2n})^{3/2} \to e^{3/2}.$$

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

    Proof. For each natural number k there is some n_k such that $|x_{n_k}| > k$ and thus $1/k > |1/x_{n_k}| \geqslant 0$. Then by the squeeze theorem $1/x_{n_k} \to 0$. ■

14. Let (x_n) be a bounded sequence and let $s := \sup\{x_n | n \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.

    Proof. By the definition of supremum for each $k \in \mathbb{N}$ there is an n_k such that $s - 1/k < x_{n_k} \leqslant s$. By the squeeze theorem $x_{n_k} \to s$. ■