# MATH 301
# Homework 8 Answer Key

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May 3, 2018

### Section 3.4

9. Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

   *Proof.* Suppose that $x_n$ does not converge to 0, then there is some $\varepsilon_0$ and a subsequence $x_{n_k}$ such that $|x_{n_k}| \geqslant \varepsilon_0 > 0$ for all $k$. This subsequence cannot have a subsequence that converges to 0 by construction. ∎

11. Suppose that $x_n \geqslant 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that $(x_n)$ converges.

   *Proof.* Note that $x_n = |(-1)^n x_n|$ and thus $\lim(x_n) = |\lim((-1)^n x_n)|$. Further, since $(-1)^n x_n$ has strictly positive and strictly negative sequences we have $\liminf((-1)^n x_n) \leqslant 0 \leqslant \limsup((-1)^n x_n)$. Because $(-1)^n x_n$ converges we have that lim inf and lim sup must coincide the limit of both $(-1)^n x_n$ and $x_n$ must be zero. ∎

17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence $(x_n)$ given by

   $$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \dots).$$

   Determine the value of $\limsup(x_n)$ and $\liminf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

   *Proof.* Note that $1 + 1/n > -1/m$ for any $n, m \in \mathbb{N}$ and further $\lim(1 + 1/n) = 1$ and $\lim(-1/n) = 0$. Let $x_{n_k}$ be an arbitrary subsequence. By construction $x_{n_k}$ must take values exclusively from $(1 + 1/n)$ and $(-1/n)$. Thus if $x_{n_k}$ does not have a tail in one of these sequences it must have a subsequence in both and thus must diverge. Consequently the set of subsequential limits is $\{1, 0\}$ and theorem 3.4.11 tells us that $\liminf(x_n) = 0$ and $\limsup(x_n) = 1$. Also since $(1 + 1/n)$ is decreasing and $(-1/n)$ is increasing we also have that $\sup\{x_n\} = 2$ and $\inf\{x_n\} = -1$. ∎

19. Show that if $(x_n)$ and $(y_n)$ are bounded sequences, then

   $$\limsup(x_n + y_n) \leqslant \limsup(x_n) + \limsup(y_n).$$

   Give an example in which the two sides are not equal.

   *Proof.* Let $v > \limsup(x_n)$ and $u > \limsup(y_n)$, then there are finitely many $n$ such that $v > x_n$ and $v > y_n$ and consequently $u + v > x_n + y_n$. Thus $y + v \geqslant \limsup(x_n + y_n)$ implying $\limsup(x_n) + \limsup(y_n) \leqslant \limsup(x_n + y_n)$.

   A counterexample to equality is the sequences $(-1)^n$ and $(-1)^{n+1}$. ∎