MATH 301
Homework 9 Answer Key

Matthew Kousoulas

May 5, 2018

Section 3.5

2. Show directly from the definition that the following are Cauchy sequences.

   (a) $\left(\dfrac{n+1}{n}\right)$,

   Proof. Let $\varepsilon > 0$ be arbitrary and then choose $N \in \mathbb{N}$ so that $N \geqslant 2/\varepsilon$, then for all $m > n > N \in \mathbb{N}$:

$$\left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\left(1 + \frac{1}{n}\right) - \left(1 + \frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leqslant \frac{1}{n} + \frac{1}{m} < \frac{2}{n} < \frac{2}{N} \leqslant \varepsilon.$$

   ■

   (b) $\left(1 + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}\right)$,

   Proof. Let $\varepsilon > 0$ be arbitrary and then choose $N \in \mathbb{N}$ so that $N \geqslant -\log_2(\varepsilon)$, then for all $m > n > N \in \mathbb{N}$:

$$\left|\sum_{i=1}^{m} \frac{1}{i!} - \sum_{i=1}^{n} \frac{1}{i!}\right| = \left|\sum_{i=n+1}^{m} \frac{1}{i!}\right| < \frac{1}{2^n} < \frac{1}{2^N} \leqslant \varepsilon.$$

   ■

5. If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but that is not a Cauchy sequence.

   Proof. First, $\lim |x_{n+1} - x_n| = \lim \left(\sqrt{n+1} - \sqrt{n}\right) = \lim \left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = 0$. However, if $m = 4n$, then $\sqrt{4n} - \sqrt{n} = \sqrt{n}$ for all n. ■

11. If $y_1 < y_2$ are arbitrary real numbers and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$ for $n > 2$, show that (y_n) is convergent. What is its limit?

   Proof. First, $|y_n - y_{n+1}| = (2/3)\,|y_n - y_{n-1}|$ making y_n a contraction and thus Cauchy. The limit is $(2/5)y_1 + (3/5)y_2$. ■

13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \geqslant 1$, show that (x_n) is a contractive sequence. What is its limit?

Proof. Note that $x_n \geqslant 2$ for all n so that:

$$|x_{n+1} - x_n| = \left|\frac{1}{x_n} - \frac{1}{x_{n-1}}\right| = \left|\frac{x_n - x_{n-1}}{x_n x_{n-1}}\right| \leqslant \frac{1}{4}|x_n - x_{n-1}|,$$

and thus Cauchy. Then:

$$\lim(x_n) = 2 + 1/\lim(x_n) \quad \text{implying} \quad \lim(x_n) = 1 + \sqrt{2}.$$

■

Section 3.6

2. Give examples of properly divergent sequences (x_n) and (y_n) with $y_n \neq 0$ for all $n \in \mathbb{N}$ such that:

   (a) (x_n/y_n) is convergent, let $x_n = n$ and $y_n = n^2$ so $x_n/y_n = 1/n$,

   (b) (x_n/y_n) is properly divergent, let $x_n = n^2$ and $y_n = n$ so $x_n/y_n = n,$.

3. Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim(x_n) = 0$ if and only if $\lim(1/x_n) = +\infty$.

   Proof. Suppose $\lim(x_n) = 0$, then for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $x_n < \varepsilon$ for all $n > N$ and consequently $1/x_n > 1/\varepsilon$ implying that $\lim(1/x_n) = +\infty$. The converse is analogous. ■

7. Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = 0$.

   (a) Show that if $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.

   Proof. For all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $x_n/y_n < \varepsilon$ and thus $0 < x_n < y_n$ for all $n > M$. Thus if $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$. ■

   (b) Show that if (y_n) is bounded, then $\lim(x_n) = 0$.

   Proof. There exists some $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{N}$, and for all ε there exists some $N \in \mathbb{N}$ such that $|x_n/M| < |x_n/y_n| < \varepsilon/M$ for all $n > N$. Consequently $|x_n| < \varepsilon$ for all $n > N$ and thus $\lim(x_n) = 0$ ■

10. Show that if $\lim(a_n/n) = L$, where $L > 0$, then $\lim(a_n) = +\infty$.

    Proof. This is a direct consequence of theorem 3.6.5. ■