**MATH301**
Final Exam 05/17/18
Total 120
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (20) True or False? For each positive integer $n$

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

**Solution**. Note that the statement holds true when $n = 1$. Assume it holds true for $n = k$, i.e. $1^3 + 2^3 + \ldots + k^3 = (1 + 2 + \ldots + k)^2$. Note that

$$\begin{aligned} 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 &= (1 + 2 + \ldots + k)^2 + (k+1)^3 \\ &= \frac{(k+1)^2 k^2}{4} + (k+1)^3 \\ &= (k+1)^2 \frac{k^2 + 4(k+1)}{4} = (k+1)^2 \frac{(k+2)^2}{4} \\ &= (1 + 2 + \ldots + k + (k+1))^2. \end{aligned}$$

Mark one and explain.
□ True □ False

2. (20) Let $A$, $B$ be countable sets of real numbers. Prove that the set

$$C = \{c \ : \ c = ab, \ a \in A, \ b \in B\}$$

is countable.

**Solution**. Since both $A$ and $B$ are countable sets, the set $A \times B$ is countable. Let $f \ : \ C \to A \times B$ be an injection defined by $f(c) = (a, b)$ so that $c = ab$ (among possibly many pairs $(a, b)$ with $c = ab$ we select just one). The set $f(C)$ is a subset of a countable set $A \times B$, hence countable. The mapping $f$ is a bijection of $C$ into $f(C)$, hence $C$ is countable.

3. (20) True or False? The sequence $x_n = n \sin n$ diverges.

**Solution**. Note that when $b-a>1$ the interval $[a,b]$ contains an integer. Let $\delta > 0$ such that $\frac{\pi}{2} - \delta > 1$. Then there is an integer $n_1 \in \left[\delta, \frac{\pi}{2}\right]$, and $\sin n_1 > \sin\delta > 0$. Analogously there is an integer $n_k \in \left[2\pi k + \delta, 2\pi k + \delta\frac{\pi}{2}\right]$, and $\sin n_k > \sin\delta > 0$. If $\lim_{k\to\infty} n_k \sin n_k$ exists, then it must be positive.

A similar argument shows that there is $m_1 \in \left[\pi + \delta, \frac{3\pi}{2}\right]$, and $\sin m_1 < -\sin\delta < 0$, and $m_k \in \left[2\pi k + \pi + \delta, 2\pi k + \frac{3\pi}{2}\right]$ with $\sin m_k < -\sin\delta < 0$. Again, if $\lim_{k\to\infty} m_k \sin m_k$ exists, then it must be negative.

The above arguments show that $x_n = n\sin n$ diverges.

Mark one and explain.

□ True □ False

4. (20) For $a > 0$ compute $\sum_{n=0}^{\infty} \frac{1}{(a+n)(a+n+1)}$, or prove that the series diverges.

**Solution**. Note that $\frac{1}{(a+n)(a+n+1)} = \left[\frac{1}{a+n} - \frac{1}{a+n+1}\right]$, and

$$\begin{aligned}
\sum_{n=0}^{k} \frac{1}{(a+n)(a+n+1)} &= \left[\frac{1}{a+0} - \frac{1}{a+1}\right] \\
&+ \left[\frac{1}{a+1} - \frac{1}{a+2}\right] \\
&+ \dots \\
&+ \left[\frac{1}{a+k} - \frac{1}{a+k+1}\right] \\
&= \frac{1}{a} - \frac{1}{a+k+1}.
\end{aligned}$$

5. (20) The set of rational numbers in $[0,1]$ is countable, hence can be considered as a sequence $\{x_n\}$. Find $\limsup\{x_n\}$.

**Solution**. Let $u_k = \sup\{x_k, x_{k+1}, \dots\}$. Since $u_k = 1$, $k = 1, 2, \dots$, and $\lim_{k\to\infty} u_k = \limsup\{x_n\}$ the result follows.

6. (20) Find $\sup\left\{1 - n - e^{-n} \ : \ n = 0, \pm 1, \pm 2, \dots\right\}$.

**Solution**. Note that when $n = 0$ one has $1 - n - e^{-n} = 0$, and for $n \geq 1$ the inequality $0 > 1 - n - e^{-n}$ holds true. When $n = -1, -2, \dots$ denote $n$ by $-k$, $k = 1, 2, \dots$, and note that $1 - n - e^{-n} = 1 - (e^k - k)$. When $k = 1$ one has $e^k - k = e - 1 > 1$. Assume $e^k - k > 1$, and note that $e^{(k+1)} - (k+1) = ee^k - k - 1 > e^k - k + e^k - 1 > e^k - k > 1$. This implies $e^k - k > 1$, $k = 1, 2, \ldots$, and $0 \ge 1 - n - e^{-n}\ :\ n = 0, \pm 1, \pm 2, \ldots$.

$\sup \left\{1 - n - e^{-n}\right\} =$

7. (20) Let $\{x_{i1}, x_{i2}, \ldots, x_{in}, \ldots\}$ be a convergent sequence with $\lim_{n\to\infty} x_{in} = x_i$. Assume that $\lim_{i\to\infty} x_i = x$. True or False? There exists a sequence $n_i$ so that $\lim_{i\to\infty} x_{in_i} = x$.

**Solution**. Le $\epsilon_1 = 1$. Select $i_1$ such that $|x_{i_1} - x| < \frac{\epsilon_1}{2}$. Select now $n_{i_1}$ such that $|x_{i_1} - x_{i_1 n_{i_1}}| < \frac{\epsilon_1}{2}$. Note that

$$|x - x_{i_1 n_{i_1}}| = |x - x_{i_1} + x_{i_1} - x_{i_1 n_{i_1}}| \le |x - x_{i_1}| + |x_{i_1} - x_{i_1 n_{i_1}}| \le \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1.$$

Repeat the argument for $\epsilon_2 = \frac{1}{2}, \epsilon_3 = \frac{1}{3}, \ldots$.

Mark one and explain.

▫ True ▫ False

8. (20) Let $\{x_n\}$ be a sequence. Define $s_n$ by $s_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$. True or False? If $\lim_{n\to\infty} s_n$ does exist, then $\lim_{n\to\infty} \frac{x_n}{n} = 0$.

**Solution**. Let $\lim_{n\to\infty} s_n = s$. Assume that $\lim_{n\to\infty} \frac{x_n}{n}$ is not 0. There is $\epsilon > 0$, and a sequence $n_k$ such that $\frac{|x_{n_k}|}{n_k} > \epsilon$. Select positive $\epsilon_1 < \frac{\epsilon}{3}$. There is $N$ such that

$$|s_n - s| < \epsilon_1, \text{ and } \frac{|s|}{n} < \epsilon_1 \text{ if } n \ge N.$$

In particular when $n_k > N$ one has

$$\begin{aligned}
\epsilon_1 &> |s_{n_k} - s| \\
&= \left|\frac{x_1 + x_2 + \ldots + x_{n_k - 1} + x_{n_k}}{n_k} - s\right| \\
&= \left|\left(\frac{x_1 + x_2 + \ldots + x_{n_k - 1}}{n_k} - s\right) + \frac{x_{n_k}}{n_k}\right| \\
&= \left|\frac{n_k - 1}{n_k}\left(\frac{x_1 + x_2 + \ldots + x_{n_k - 1}}{n_k - 1} - s\right) - \frac{1}{n_k}s + \frac{x_{n_k}}{n_k}\right| \\
&\ge \epsilon - \frac{n_k - 1}{n_k}\epsilon_1 - \frac{|s|}{n_k} \\
&\ge \epsilon - \epsilon_1 - \epsilon_1 > \frac{1}{3}\epsilon > \epsilon_1.
\end{aligned}$$

This contradiction completes the proof.

Mark one and explain.

□ True □ False

9. (20) Let $\{x_n\}$ be a sequence of positive numbers. Define $s_n$ by $s_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$. True or False? If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} s_n = x$.

**Solution**. First note that

$$|x - s_n| = \frac{1}{n}|nx - (x_1 + \ldots + x_n)| \le \frac{|x - x_1|}{n} + \frac{|x - x_2|}{n} + \ldots + \frac{|x - x_n|}{n}.$$

Let $\epsilon > 0$, then there is $N$ so that $|x - x_k| > \frac{\epsilon}{2}$ whenever $k \ge N$. So if $k = K + N$, then

$$|x - s_{K+N}| \le \frac{|x - x_1| + \ldots + |x - x_{N-1}|}{K + N} + \frac{|x - x_N| + \ldots + |x - x_{N+K}|}{K + N}$$

(a) If $K > \frac{2}{\epsilon}(|x - x_1| + \ldots + |x - x_{N-1}|)$, then $\frac{|x - x_1| + \ldots + |x - x_{N-1}|}{K + N} < \frac{\epsilon}{2}$.

(b) $\frac{|x - x_N| + \ldots + |x - x_{N+K}|}{K + N} \le \frac{\epsilon}{2}\frac{K + 1}{K + N} < \frac{\epsilon}{2}$.

This shows $\lim_{n\to\infty} s_n = x$.

Mark one and explain.

□ True □ False

10. (20) True or False? Let $a$ be a real number, and $B$ is a set of real numbers such that $a \notin B$. If every open interval containing $a$ contains a point of $B$, then for every $\epsilon > 0$ the interval $(a - \epsilon, a + \epsilon)$ contains infinitely many points of $B$.

**Solution**. Let $\epsilon_1 = \epsilon$, and $b_1 \in B$ so that $|a - b_1| < \epsilon_1$. If $\epsilon_k$ and $b_k$ are already defined, then define $\epsilon_{k+1} = \min\left\{\frac{\epsilon_k}{2}, \left|\frac{a - b_k}{2}\right|\right\}$, and denote an alement of $B$ located in $(a - \epsilon_{k+1}, a + \epsilon_{k+1})$ by $b_{k+1}$. The sequence $\{b_k\}$ is located in $(a - \epsilon, a + \epsilon)$.

Mark one and explain.

□ True □ False

11. (20) True or False? Let $\{x_n\}$ be a bounded sequence, and $b = \limsup x_n$. For each $\epsilon > 0$ the set of numbers $\{x_k : b + \epsilon < x_k\}$ is finite.

**Solution**. Since $\{x_n\}$ is a bounded sequence $b = \limsup x_n$ is a finite number. Assume there are infinitely many $x_k$ elements of the sequence $\{x_n\}$ so that $b + \epsilon < x_k$. If $u_m = \sup\{x_m, x_{m+1}, \ldots\}$, then $u_m \ge b + \epsilon$, and $b = \limsup x_n = \lim u_m \ge b + \epsilon$. This contradiction completes the proof.

Mark one and explain.

□ True □ False