**MATH301**
quiz #1, 03/01/18
Total 120
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (20) True or False? The set irrational numbers $I$ is countable.

   **Solution**. By contradiction. If $I$ is countable, then $\mathbf{R} = I \cup Q$ is countable. This contradiction shows that the assumption is false.

   Mark one and explain.
   ▫ True ▫ False

2. (20) Solve the inequality $|2x| \le |x^2 + 1|$.

**Solution**. For $x \ge 0$ the inequality becomes $2x \le x^2 + 1$. The inequality hold for each $x$. For $x < 0$ the inequality becomes $-2x \le x^2 + 1$. The inequality hold for each $x$.

$x =$

3. (20) True or False? For each $k = 1, 2, \dots$ one has $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.

**Solution**. We shal use mathematical induction. When $n = 1$ the statement is correct. Assume the statement holds for $n = k$. Note that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Since $\sqrt{k}\sqrt{k+1} > k$ one has $\sqrt{k}\sqrt{k+1} + 1 > k + 1$, and $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$.

Mark one and explain.

□ True □ False

4. (20) Let $x$ and $y$ be two positive real numbers such that $1 < y - x$. Show that there is $n \in \mathbf{N}$ such that $x \leq n \leq y$.

**Solution**. First note that if $y \in \mathbf{N}$, than we can set $n = y$. From now on we assume that $y \notin \mathbf{N}$. Let $\mathbf{N}_y = \{z \ : \ z \in \mathbf{N} \text{ and } y < z\}$, and $l = \inf \mathbf{N}_y$. One has $l \in \mathbf{N}$ and $y < l$. Note that $x < l - 1 < y$. We can now set $n = l - 1$.

5. (20) Let $f(x) = x + \frac{1}{x}$, and $D = \{x \, : \, x > 0\}$. Find $\inf f(D)$.

**Solution**. Note that $0 \le (x-1)^2 = x^2 - 2x + 1$, and $2x \le x^2 + 1$. This yields $2 \le x + \frac{1}{x}$. Since $f(1) = 2$ one has $\inf f(D) = 2$.

$\inf f(D) =$

6. (20) Let $S = \{|x| \ : \ x = n\sqrt{2} + m, \ n, m \in \mathbf{Z}\}$. Find $\inf S$.

**Solution**. Since $0 \le |x|$ for every $x$ one has $0 \le \inf S$. We next show that $\inf S < \frac{1}{2}$. Let $I_1 = \left\{x \ : \ 0 < x < \frac{1}{2}\right\}$, and $I_2 = \left\{x \ : \ \frac{1}{2} < x < 1\right\}$. For an integer $n$ we denote by $n + I_k = \{x \ : \ n + y, \ y \in I_k\}$, $k = 1, 2$. We note that $\sqrt{2} \in 1 + I_1$, and for every $n \in \mathbf{Z}$ there is $n_k \in \mathbf{Z}$ and an index $j_n \in \{1, 2\}$ such that $n\sqrt{2} \in n_k + I_{j_n}$. So, for example,

| $n$ | number | interval | $j_n$ |
|---|---|---|---|
| 1 | $\sqrt{2}$ | $1 + I_1$ | 1 |
| 2 | $2\sqrt{2}$ | $2 + I_2$ | 2 |
| 3 | $3\sqrt{2}$ | $4 + I_1$ | 1 |

We note that $\sqrt{2} - 1 \in I_1$, and $3\sqrt{2} - 4 \in I_1$. That is

$$0 < \sqrt{2} - 1 < \frac{1}{2}, \text{ and } 0 < 3\sqrt{2} - 4 < \frac{1}{2}.$$

Hence $\frac{1}{2} > \left|(\sqrt{2} - 1) - (3\sqrt{2} - 4)\right| = \left|2\sqrt{2} - 3\right|$.

Repetition of this construction with intervals $I_j = \left\{x \ : \ \frac{j-1}{n} < x < \frac{j}{n}\right\}$, $j = 1, \ldots, n$ shows that $\inf S < \frac{1}{n}$.

$\inf S =$